SOLID
GEOMETRY

PALMER AND TAYLOR
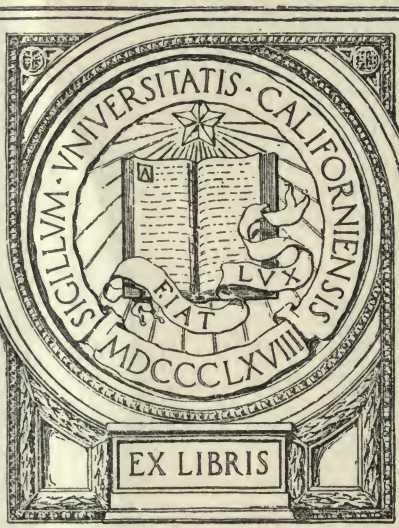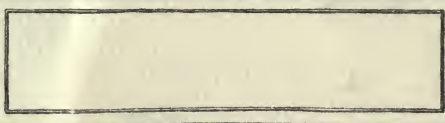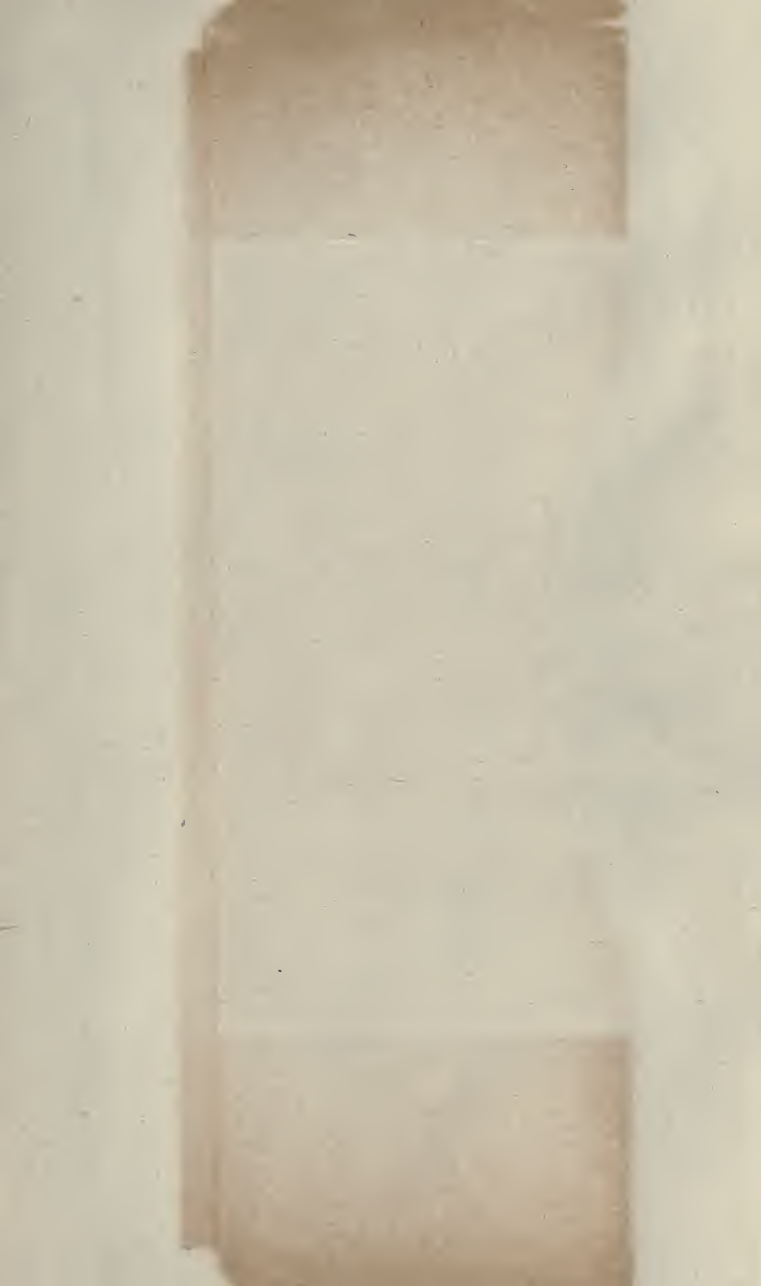
SOLID GEOMETRY

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PREFACE

The purpose of this book is to present the subject of solid geometry so as to emphasize some of its applications as well as to give a thorough training in logical reasoning. To accomplish this the authors have continued the same method of presentation that characterizes their Plane Geometry.

The aim of the authors in the preparation of this text has been to present the subject in a logical manner that is comprehensible to the youthful mind, and to vitalize the subject-matter — making it both interesting and useful through a wide range of practical applications.

Some of the features peculiar to the Solid Geometry are outlined in the following paragraphs:

1. In Chapter VI, the theorems on the relations of lines to planes and of planes to planes are carefully grouped so as to make them more easily comprehended by the student.

2. Distinct advantage is gained by combining certain related subjects, thus securing brevity and simplicity of treatment. Prisms are combined with cylinders, pyramids with cones, and polyedral angles with spherical polygons.

3. In the theorems regarding the areas and volumes of solids, and requiring a use of limits, great care is taken to make the treatment logical and still keep it within the grasp of the average student.

4. The large number and great variety of exercises are carefully distributed, and range from some quite elementary to others of considerable difficulty. This enables the teacher to adapt the book to any group of students, whether in a classical or a technical high school. The abstract exercises give mental training and application of the basic theorems, while the practical exercises are used to correlate geometric facts with real life.

Many of the exercises involve an application of arithmetic and algebra to geometry.

In addition to the four points outlined above, attention is called to the following features:

While many theorems are proved in full, the proof of others is given only in part. In still others the method of proof is suggested only or the work is left entirely to the student. Thus a middle course is adopted in the use of the suggestive method.

The Prismatoid Formula, one of the most powerful theorems of solid geometry, is proved in a simple direct manner, and is followed by various applications. However, this may be omitted and in no way interfere with the proofs of later theorems.

The more important theorems are printed in heavy type. This is in accordance with the recommendation of the Committee of Fifteen, whose report, as well as various other committee reports, have been given careful consideration in the preparation of the entire work.

For convenience, a combination ruler and protractor accompanies each book.

Acknowledgment is due to the McGraw-Hill Book Co., Inc., for permission to use exercises from Palmer's *Practical Mathematics*, in which are gathered numerous exercises from the author's many years of experience in teaching practical students.

<div style="text-align: right">

C. I. Palmer.

D. P. Taylor.

</div>

Chicago, September, 1918.

CONTENTS

SOLID GEOMETRY

INTRODUCTION

528. Solid geometry may be thought of as an extension of plane geometry, with which it has much in common. The figures of plane geometry lie in a plane; while in solid geometry we consider the properties of figures that do not lie entirely in one plane. Such figures are sometimes called **three-dimensional figures.**

529. Space Ideas. We gain ideas concerning space and objects in space through the senses of touch, sight, and hearing. Mainly through touch and sight, we determine the size and shape of an object.

530. Difficulties. Most beginners in the study of solid geometry have difficulty in visualizing, or forming a mental image, of a three-dimensional figure from a worded description or from a drawing in a plane.
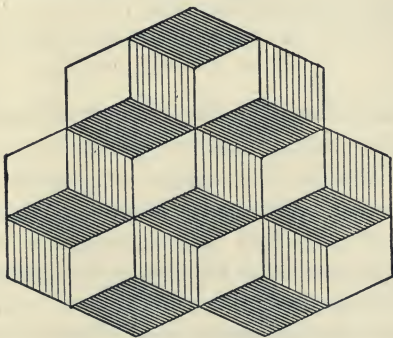
Even the same picture may present to the minds of two persons quite different images. For instance, in the figure, one person may see six blocks while another will see seven.

A little practice, however, in visualizing the blocks enables one at will to see six or seven blocks.

Frequently a student does not understand a proposition or a proof because he does not visualize the figure properly. Lines may appear to extend in a different direction than was intended, or a plane may seem to lie in front of the page when it was intended that it should appear back of the page.

Whatever device may be necessary for a clear image should be used; but care should be taken not to become too dependent upon models. A very important part of the training from the study of solid geometry is that it trains in visualizing three-dimensional figures from a description or a drawing.

531. Devices to Assist in Imaging. Cardboard models may be formed as described in § 753.

Planes can be made to intersect by cutting two pieces of cardboard half in two and fitting together along the cuts.

Cardboard can be used in connection with sharp pointed wires, such as hatpins. § 568.

Hatpins or sharp wires held together by small corks can be used to form open models. § 571.

Various figures can be formed with the hatpins and a piece of soft board.

532. Representation of Solid Geometry Figures. In solid geometry, the figures are usually represented on paper or on the blackboard, that is, in a plane. It is well to have some idea of how to make the figures appear solid, and to "stand out" as desired. While it is not thought best to enter into a discussion of perspective, a few general suggestions that are of assistance will be given.
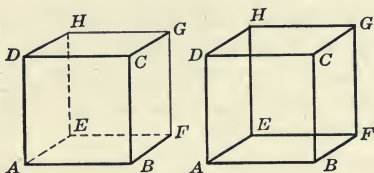
The figures may be supposed to be viewed from a point a little above, and either directly in front or a little to the right or left. Parts of the figure that are rectangles, in general, appear as parallelograms in the drawing. §§ 663-5, 671, 678. Polygons appear as polygons, but reduced in one direc-

tion. §§ 660, 749. Parts that are circles appear as ellipses,
which become narrower as the eye of the observer is nearer
to the plane in which the circle lies. §§ 689, 692, 864.

Planes are supposed to be opaque, and all lines covered by
them should be left out, or, if it is desired to insert them,
they should be dashed lines.

Two lines that intersect in the drawing do not necessarily
intersect in the solid figure. Lines are not usually of the
same length as they are drawn. The size of an angle may be
quite different from the size in the drawing.

In the accompanying drawings representing cubes, the one at the left
appears to have the face *ABCD* in front and can hardly be imaged other-

wise. While the one at the right can easily be seen as if viewed from
a point a little above and to the right, and so having the face *ABCD*
in front; or it may be visualized as viewed from a point a little below
and to the left, and so having the face *EFGH* in front.

533. The Use of Plane Geometry in Solid Geometry. All
of the plane geometry is at the disposal of the student in
solid geometry; but, in applying the facts of plane geometry,
great care must be taken not to use them except where they
will hold.

Facts of plane geometry that hold without reference to any
plane are the axioms and most definitions, theorems stating
that figures are congruent, and all the theorems where it is
evident that it does not matter if the figures are not in the
same plane.

Theorems of plane geometry that hold because the nature
of the figure requires that it lie in a plane are those concern-

ing a triangle, two parallel lines, a parallelogram, a circle, and all other theorems where the conditions stated determine a plane in which the parts must lie.

In all other cases, the theorems of plane geometry can be applied only after all the parts concerned are shown to lie in one plane. Under these will fall many theorems concerning parallels, perpendiculars, circles, and polygons.

EXERCISES

1. Draw the figure of § 603, omitting the lines *DE*, *EF*, and *DF*. Can you visualize the figure as otherwise than in one plane?

2. Draw the figure of § 572, making the side lines of plane *P* full. Compare with the figure of the text.

3. Draw the figure of § 628, making all lines full and of the same thickness. Compare with the figure of the text.

4. Draw the figure of § 671, making all lines of the same thickness. Does your figure appear as if viewed from below? If not, make such lines heavy as will make it appear so.

5. Draw a figure like the one at the right in § 722, but change the lines so that it will appear as if viewed from below.

6. Draw a figure of a sphere as in § 787, but viewed from a point on a level with the center of the sphere.

STRAIGHT LINES AND PLANES

534. Surface. A boundary of a portion of space is called a **surface.** A surface may be limited or unlimited in extent. It has two dimensions.

535. Plane. A surface such that a straight line joining any two of its points, lies wholly in the surface, is called a **plane surface** or, simply, a **plane.**

Since a straight line may be unlimited in length, this definition implies that a plane may be unlimited in extent, for otherwise the straight line could not lie wholly in the surface.

It also follows from the definition that a straight line can intersect a plane in but one point.

536. Representing a Plane. In drawing a geometric figure a portion of the plane is conveniently represented as a rectangle seen obliquely. In order to make the plane appear to "stand out," the front edge is frequently drawn longer than the back edge. Making the edges on the front and one end heavy and shading help one to image a plane as it is intended it should appear.

537. Reading a Plane. A plane may be read by a single letter, by letters at opposite vertices of the rectangle that represents it, or by the letters at all the vertices of the rectangle. Thus, in the figure, the planes may be read "plane *M*," "plane *AC*," or "plane *ABCD*."

EXERCISES

1. Hold two pencils so that they determine a plane. So that they do not determine a plane.

2. In making a kite, two straight sticks AB and CD are fastened together at P. Show that paper stretched over these sticks lies in a plane.
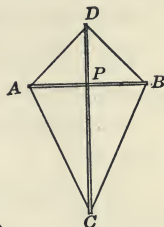
3. Show that a rubber cord fastened at both ends and stretched by grasping it at any other point, lies in one plane.

4. From a given point lines are drawn to points on a given line; show that all these lines lie in the same plane.

5. What is the locus of all lines that pass through a given point and intersect a given line not containing the point?

6. Show that, if a rubber band is stretched by three hooks in any position, all parts of the band lie in one plane.

7. Do four points necessarily lie in a plane? When do they? Give illustrations by using points in the classroom.

8. Why is a tripod used to support a camera or a surveyor's transit?

9. Why do the four legs of a chair sometimes not all rest upon a floor?

10. How many planes are determined by four points not all lying in the same plane?

11. Given five points four of which are in the same plane. How many planes do they determine?

12. How many planes are determined by three concurrent lines that do not all lie in the same plane?

13. How does a carpenter determine whether a floor is a plane?
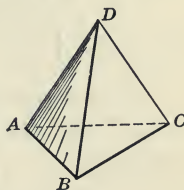
14. Why does a mason use a trowel with long straight edges when "truing up" a wall?

15. How many planes are determined by four lines all meeting in a point but no three of which lie in the same plane?

16. Must a triangle lie in a plane? Must a parallelogram? Must a trapezoid? Must every quadrilateral?

17. Is the following a complete definition for a circle: A curved line every point of which is equally distant from a fixed point called the center is a circle? Why?

18. Through a point in space (1) draw a line parallel to a given line; (2) draw a line perpendicular to a given line.

RELATIVE POSITIONS OF LINES AND PLANES

545. Definitions. Collinear means lying in the same line. **Coplanar** means lying in the same plane.

546. Relations of Two Lines. From the study of plane geometry and §§ 542, 543, it follows:

(1) That two coplanar lines may *coincide, intersect,* or *be parallel.*

(2) That two non-coplanar lines are *neither intersecting nor parallel.*

547. Relations of a Line and a Plane. A straight line may have three positions relative to a plane:

(1) *It may lie in the plane.*

(2) *It may intersect the plane.*

(3) *It may be parallel to the plane.*

A line **lies in a plane** if all of its points are in common with the plane.

A line **intersects a plane** if it has only one point in common with the plane. The point in common is called the **foot** of the line.

A line is **parallel to a plane,** and the plane is parallel to the line, if they have no point in common, that is, if they do not meet.

548. Relations of Two Planes. Two planes may have three relative positions:

(1) *They may coincide.*

(2) *They may intersect.*

(3) *They may be parallel.*

Two planes are said to **coincide** if they have the same determining conditions.

In plane geometry, the relative positions of points and lines are few and so the treatment of the subject is quite informal. In solid geometry, the study is made of points, lines, and planes, and their relative positions are many. It is advisable then, to make a more formal study of the subject.

INTERSECTING PLANES

549. Definition. The **intersection of two surfaces** consists of all points common to the two surfaces, and no other points; that is, it is the locus of all points common to the two surfaces.

550. Axiom. *If two planes have one point in common, they also have another point in common.*

551. Theorem. *If two planes intersect, their intersection is a straight line.*
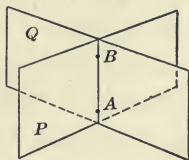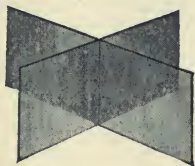
Given two intersecting planes P and Q.

To prove that their intersection is a straight line.

Proof. Let A and B be two points common to both planes.   § 550

The straight line determined by A and B lies in the plane P and also in Q.   § 535

Then AB is common to the two planes.   Why?

Moreover, no point not in AB is common to the two planes for then they would coincide.   § 540

∴ the intersection of planes P and Q is a straight line.

EXERCISES

1. Can three planes intersect (1) in one line? (2) in two lines? (3) in three lines? (4) in more than three lines? Explain.

2. If a paper is folded and creased, why does it form a straightedge along the crease?

3. From a point external to two non-coplanar lines only one line can be drawn that will cut the two lines.

552. Theorem. *If three planes, not passing through the sam line, intersect each other, their three lines of intersection eithe meet in a point or are parallel each to each.*

Given the three planes AD, CF, and AF not passing through the same line, and intersecting in lines AB, CD, and EF.

To prove that AB, CD, and EF either meet in a point or are parallel each to each.

Proof. Consider the two lines of intersection lying in any plane. (1) These lines intersect, or (2) they are parallel.

(1) Suppose AB and CD intersect at P.

Then P lies in plane AF and in plane CF, and hence in their intersection EF. § 549

∴ AB, CD, and EF meet in point P.

(2) Suppose $AB \parallel CD$.

Then $EF \parallel CD$ or intersects CD.

Suppose EF intersects CD, then AB intersects CD by (1). But this is impossible, for $AB \parallel CD$.

Hence $EF \parallel CD$.

Similarly it can be proved that $EF \parallel AB$.

∴ AB, CD, and EF are parallel each to each.

EXERCISES

1. Three planes that do not contain the same straight line can have but one point in common.

2. Explain the meaning of the expression, "Two planes determine a straight line." Is it always true that two planes determine a straight line? Is the analogous statement, "Two lines determine a point," of plane geometry always true? Explain.

LINES AND PLANES, PARALLEL

553. Theorem. *Two straight lines that are parallel to a third straight line are parallel to each other.*

Given lines AB and CD, each parallel to EF.

To prove $AB \parallel CD$.

Proof. Either AB is parallel to CD or it is not.

AB and EF determine plane AF.	Why?
CD and EF determine plane CF.	Why?
AB and C determine plane BC.	Why?

If AB is not parallel to CD, plane BC will intersect plane CF in some line CG other than CD.   § 551

Then	$CG \parallel EF$.	§ 552 (2)
But	$CD \parallel EF$.	Given
Hence	CD and CG coincide.	§ 118
	$\therefore AB \parallel CD$.	Why?

554. Theorem. *A plane containing one, and but one, of two parallel lines is parallel to the other.*

Given two parallel lines AB and CD, and plane P containing CD but not AB.

To prove plane $P \parallel AB$.

Suggestion. AB and CD determine a plane. Suppose AB not parallel to plane P, and show that then AB would not be parallel to CD.
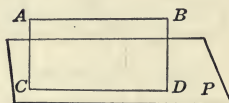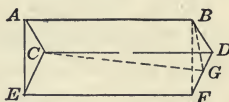
EXERCISES

1. As a door is opened, show that all the positions of its outer edge are parallel to each other.

2. Prove that the lines connecting in order the middle points of the sides of a quadrilateral in space form a parallelogram.

By a "quadrilateral in space" is meant a quadrilateral not all of whose sides lie in the same plane. It is often called a **skew** or **gauche quadrilateral**.

555. Theorem. *If a line is parallel to a plane, it is parallel to the intersection of that plane with any plane through the line.*

556. Theorem. *A line is in a plane if it has a point in the plane, and the line and the plane are both parallel to a second line.*

The two parallel lines determine a plane intersecting the first plane.

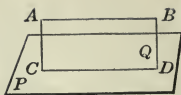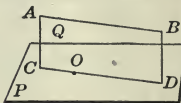**557. Theorem.** *If a line is parallel to the intersection of two planes, it is parallel to each of the planes.*

558. Theorem. *A line parallel to each of two intersecting planes is parallel to their intersection.*

A line through a point in the intersection of the two planes and parallel to the given line must lie in both planes by § 556.

559. Problem. *Through a given line to pass a plane parallel to any other straight line not intersecting the given line.*

(1) How many planes can be passed through the given line and parallel to the other line if the given line is parallel to the other line? §§ 541, 554.

(2) If the given line is not parallel to the other, construct a line parallel to the given line and through a point in the other. Then the intersecting lines determine a plane parallel to the given line. 557

560. Problem. *Through a point to pass a plane parallel to each of two given non-coplanar lines.*

Given AB and CD, the two lines, and P, the point.

Construction. Through P construct lines parallel to both AB and CD. These lines determine the plane parallel to AB and CD.
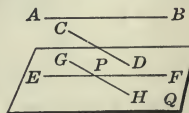
Give proof.

EXERCISES

1. If a line in one of two intersecting planes is parallel to the other plane, it is parallel to their intersection.

2. If a line and a plane are parallel to the same line, they are parallel to each other, or the line lies in the plane. 554

561. Definition. Two planes are parallel if they have no points in common, that is, if they do not meet however far they may be produced.

562. Theorem. *If a plane intersects two parallel planes, the lines of intersection are parallel.*

Given plane *PQ* intersecting the two parallel planes *MN* and *RS* in *AB* and *CD*, respectively.

To prove *AB* ∥ *CD*.

Suggestion. Since *AB* and *CD* are in the same plane *PQ*, they are parallel or intersect. Suppose that they intersect. What follows?

563. Theorem. *Parallel lines intercepted between parallel planes are equal.*   § 158

564. Theorem. *A line is in a plane if it has a point in the plane and if the line and plane are both parallel to a second plane.*

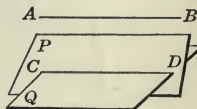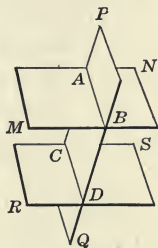Pass a plane through the line and intersecting the two planes. Then apply § 118.

EXERCISES

1. If a line and a plane are parallel to the same plane, they are parallel to each other, or the line lies in the plane.

2. State the converse of the theorem of § 562. Is it true?

3. A board is cut across on a slant. What is the shape of the cut section?

565. Theorem. *If two intersecting straight lines are parallel to a plane, the plane determined by these lines is parallel to that plane.*

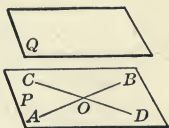**Given** the intersecting lines *AB* and *CD*, each parallel to plane *Q*, and the plane *P* determined by *AB* and *CD*.

To prove *P* ∥ *Q*.

Proof. Either *P* is or is not parallel to *Q*.

If *P* is not parallel to *Q*, it will intersect it in a line parallel to both *AB* and *CD*.   § 555

But this is impossible.   § 118

$$\therefore P \parallel Q.$$

566. Theorem. *Through a given point one plane, and only one can be passed parallel to any given plane not containing the point.*

Pass a plane through the given point and intersecting the given plane. Through the point and in this plane draw a line parallel to the intersection of the two planes. This line is parallel to the given plane by § 554.

In a similar manner draw a second line through the given point, and apply § 565.

567. Theorem. *If two angles, not in the same plane, have their sides parallel, right side to right side and left side to left side, the angles are equal and their planes are parallel.*

Given $\angle x$ in plane P and $\angle z$ in plane Q, having $AB \parallel DE$ and $CB \parallel FE$.

To prove $\angle x = \angle z$, and $P \parallel Q$.

Proof. Take $BA = ED$, and $BC = EF$, and draw lines AD, CF, BE, AC, and DF.
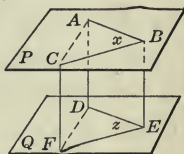
Both AE and CE are parallelograms.	Why?
Hence both AD and CF are equal and parallel to BE.	Why?
Then $\qquad$ AD and CF are equal and parallel.	Why?
Therefore $\qquad$ $ADFC$ is a parallelogram.	§ 160
And $\qquad$ $AC = DF$.	Why?
Then $\qquad$ $\triangle ABC \cong \triangle DEF$.	§ 113
$\qquad$ $\therefore \angle x = \angle z$.	Why?
Also $\qquad$ $AB \parallel Q$, and $CB \parallel Q$.	§ 554
$\qquad$ $\therefore P \parallel Q$.	§ 565

EXERCISES

1. What is the locus of a line that passes through a given point, and is parallel to a given plane?

2. If two congruent parallelograms lie in parallel planes and have their sides respectively parallel, how many planes are determined in using their sides in all possible combinations?

3. The four lines in which two parallel planes intersect two other parallel planes are parallel.

4. If two planes are parallel to the same line, their intersections with any plane through the line are parallel to each other.

568. Theorem. *If two straight lines are cut by three parallel planes, their corresponding segments are proportional.*
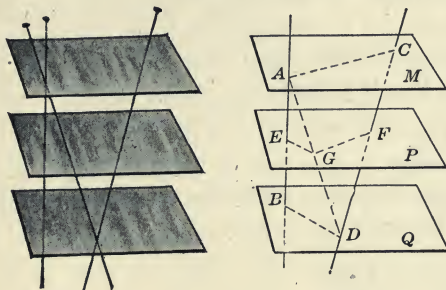
Given AB and CD two straight lines intersected by three parallel planes M, P, and Q, in the points A, E, and B, and C, F, and D respectively.

To prove $\dfrac{AE}{EB} = \dfrac{CF}{FD}$.

Proof. Draw AD intersecting plane P in G.

Let the plane determined by AB and AD intersect plane P in EG and plane Q in BD.

Then $EG \parallel BD$.   § 562

Similarly, if AD and CD determine a plane intersecting plane M in AC and plane P in GF, $GF \parallel AC$.   Why?

Hence, $\dfrac{AE}{EB} = \dfrac{AG}{GD}$, and $\dfrac{CF}{FD} = \dfrac{AG}{GD}$.   § 411

$$\therefore \frac{AE}{EB} = \frac{CF}{FD}.$$ Why?

569. Theorem. *If two straight lines are cut by a series of parallel planes, the corresponding segments are proportional.*

EXERCISE. A mine shaft passes for 180 ft. on a slant through two layers of rock. One, a layer of hard rock, is 72 ft. thick; and the other, a layer of soft rock, is 84 ft. thick. If it costs $4 per linear foot in the hard rock and $3 in the soft to make the shaft, find the cost of digging the shaft.

LINES AND PLANES, PERPENDICULAR

570. Definition. A straight line is **perpendicular to a plane** if perpendicular to all straight lines in the plane and passing through its foot. The plane is also **perpendicular to** the line.

A straight line is **oblique to a plane** when it meets the plane but is not perpendicular to it.

571. Theorem. *If a line is perpendicular to each of two intersecting lines at their point of intersection, it is perpendicular to the plane determined by these lines.*
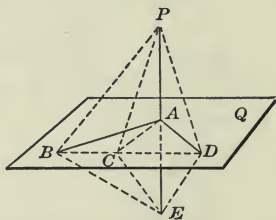
Given $AP \perp AB$ and AD at their point of intersection A, and plane Q determined by AB and AD.

To prove AP perpendicular to the plane Q.

Proof. In plane Q draw BD, any line not passing through A, and through A draw any line AC meeting BD in C.

Prolong PA to E, making $AE = AP$; and draw BP, CP, DP, BE, CE, and DE.

Then AB and AD are perpendicular bisectors of PE.		Why?
Therefore	$BP = BE$, and $DP = DE$.	§ 205
Then	$\triangle BPD \cong \triangle BED$.	§ 113
And	$\angle CBP = \angle CBE$.	Why?
Further	$\triangle CBP \cong \triangle CBE$.	§ 247
And therefore	$CP = CE$.	Why?

Hence $AC \perp PE$, *i.e.*, $AP \perp AC$, any line in plane Q and passing through its foot.   Why?

$\therefore AP$ is perpendicular to the plane Q.   § 570

572. Theorem. *All the perpendiculars that can be drawn to a straight line at a given point in the line, lie in a plane perpendicular to the line at the given point.*

Given DC any line $\perp AB$ at C, and plane $Q \perp AB$ at C.

To prove that all lines perpendicular to AB at C lie in plane Q.

Proof. DC and AB determine a plane P that intersects plane Q in EC.

Then	$EC \perp AB$.	§ 570
But	$DC \perp AB$.	Given
Therefore DC coincides with EC.		Why?
And DC lies in plane Q.		

$\therefore$ all lines $\perp AB$ at C lie in plane Q.

573. Theorem. *Through a given point one plane can be passed perpendicular to a given straight line, and only one.*

If the given point is in the given line, two perpendiculars to the line determine a plane perpendicular to the given line. If the given point is outside the given line, construct a perpendicular from the point to the line, and at its foot construct another perpendicular to the given line. These determine the perpendicular plane.

To show that only one plane can be drawn in each case, suppose that there are two planes perpendicular and pass a plane through the given line and intersecting the two planes in two lines. What follows?

EXERCISES

1. Show how a perpendicular to a plane can be determined by means of two carpenter's squares.

2. Each of three concurrent lines is perpendicular to each of the other two. Prove that each is perpendicular to the plane of the other two.

3. If a book partly open stands on a table, is the back of the book perpendicular to the table top?

4. One side of a right angle revolved about the other side as an axis generates a plane perpendicular to this other side.

5. Do the hands of a clock revolve in a plane? Why?
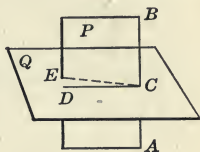
574. Theorem. *From a given point outside a plane one line perpendicular to the plane, and only one, can be drawn.*

Given plane Q and point A outside Q.

To prove one line perpendicular to Q from A, and only one, can be drawn.
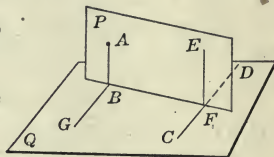
Proof. In plane Q draw any line CD. Through A pass a plane $P \perp CD$. § 573

Let BF be the intersection of P with Q.

Construct $AB \perp BF$, and $EF \parallel AB$. §§ 312, 117

In Q, construct $GB \parallel CF$. § 117

Then $\qquad\qquad CF \perp EF$. Why?

And $\qquad\qquad \angle ABG = \angle EFC$. § 567

Hence $\qquad\qquad AB \perp GB$. Why?

Therefore $\qquad\qquad AB \perp$ plane Q. § 571

Only one perpendicular can be drawn, for if two could be drawn, they would both be perpendicular to the intersection of their plane with Q, which is impossible. Why?

575. Theorem. *At a point in a plane one perpendicular can be drawn to the plane, and only one.*

Let O be the point in plane Q. Draw AB, any line in Q through O. Construct plane $P \perp AB$ through O. In P construct $OE \perp CD$. Complete and give proof.
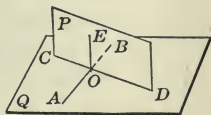
576. Theorem. *The perpendicular to a plane is the shortest line from a point to a plane.*

Apply § 185.

577. Definition. The length of the perpendicular from a point to a plane is called the **distance from the point to the plane.**

The **distance between two parallel planes** is the length of the segment of a common perpendicular lying between them.

EXERCISES

1. What is the locus of all points equidistant from a circle?

2. What is the locus of all points equidistant from the vertices of a triangle?

3. What is the locus of all points equidistant from two given points?

4. Find the length of the locus in a plane of all points 10 in. from a point 6 in. from the plane.

5. With a pair of compasses opened 13 in. show how a circle of 12 in. radius can be drawn.

6. A perpendicular smokestack is braced by three guy wires attached at a point 90 ft. from the ground. If these wires are 150 ft. long and meet the ground at the vertices of an equilateral triangle, find the length of one side of the triangle.

**7.** Could a smokestack be fully braced with three guy wires? With two? Why?

8. If from the foot of a perpendicular to a plane a line is drawn perpendicular to any given line in the plane, the line joining the point of intersection to any point in the perpendicular to the plane is perpendicular to the given line in the plane.

578. Theorem. *Two planes perpendicular to the same straight line are parallel; and conversely, if one of two parallel planes is perpendicular to a straight line, the other is also.*

Given planes P and $Q \perp AB$.

To prove $P \parallel Q$.

Outline of proof. If the planes were not parallel, they would meet. Two perpendiculars could then be drawn to the line from a point in their intersection.
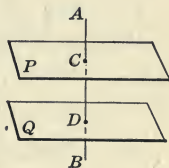
CONVERSELY:

Given plane P parallel to plane Q, and $P \perp AB$.

To prove $Q \perp AB$.

Outline of proof. AB is perpendicular to any line in P through its foot. Draw two such lines.

The planes determined by these lines and AB will intersect Q in lines parallel respectively to the lines in P, and, therefore, will be perpendicular to AB.

579. Theorem. *If two planes are parallel to a third plane, they are parallel to each other.*

Proof similar to that of § 124.

580. Theorem. *If one of two parallel lines is perpendicular to a plane, the other is also; and conversely, two straight lines perpendicular to the same plane are parallel.*

Given $AB \parallel CD$, and AB perpendicular to plane P.

To prove $CD \perp P$.

Proof. Through D draw any line DF in plane P.
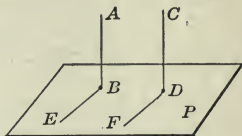
Through B draw BE in plane P and parallel to DF.

Then $\angle ABE = \angle CDF$.   § 567

But $\angle ABE$ is a rt. $\angle$.   Why?

Therefore $\angle CDF$ is a rt. $\angle$ and $CD \perp DF$, any line in plane P through D.

$$\therefore CD \perp P. \qquad\qquad \text{Why?}$$

CONVERSELY:

Given $AB \perp$ plane P, and $CD \perp$ plane P.

To prove $AB \parallel CD$.

Outline of proof. Through D draw $ED \parallel AB$. Then $ED \perp P$, and therefore coincides with CD by § 575.
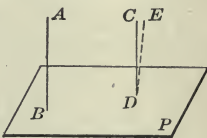
581. Theorem. *Two parallel planes are everywhere the same distance apart.*

Choose any two points in one plane and draw perpendiculars to the other plane, § 574. Then these lines are perpendicular to the first plane, § 578. Further these two lines are parallel, § 580, and determine a plane intersecting the two parallel planes in parallel lines, § 562. Then apply § 191.

EXERCISES

1. Prove the theorem of § 553 by § 580.
2. A line cannot be perpendicular to each of two intersecting planes.

PROJECTIONS

582. Definitions. The **projection of a point** upon a plane is the foot of the perpendicular from the point to the plane.

The **projection of a line** upon a plane is the locus of the projections of all points of the line upon the plane.

Thus B is the projection of point A upon plane Q, and EF is the projection of line CD.

583. Theorem. *The projection upon a plane of a straight line that is not perpendicular to the plane is a straight line.*

Given AB, a straight line not $\perp$ plane Q.

To prove that the projection of AB upon Q is a straight line.

Proof. From any point in AB, as E, draw $EF \perp Q$. § 574
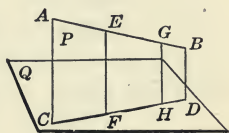
Plane P, determined by AB and EF, intersects Q in CD. § 551

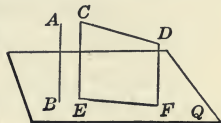From G, any other point in AB, draw $GH \perp Q$. § 574

Then $GH \parallel EF$ and lies in P. Why?

Therefore its foot H lies in CD. Why?

And the projections of all points of AB lie in CD.

Furthermore, a perpendicular to Q at any point in CD will intersect AB. Why?

Then every point in CD is the projection of a point in AB.

∴ the projection of AB upon Q is a straight line.

584. Theorem. *Of all oblique lines drawn from a point to a plane:*

(1) *Those that have equal projections are equal.*

(2) *Those that have unequal projections are unequal, and the one having the greater projection is the greater.*

Another statement of this theorem is: Of all oblique lines, drawn to a plane from a point in a perpendicular to the plane, those that cut off equal distances from the foot of the perpendicular are equal, and of those that cut off unequal distances from the foot of the perpendicular, the more remote is the greater.

585. Theorem. *The acute angle that a straight line makes with its projection upon a plane is the least angle that it makes with any line passing through its foot and lying in the plane.*

Given the straight line AB meeting plane Q at B, its projection CB in Q, and DB, any other line through B and in Q.

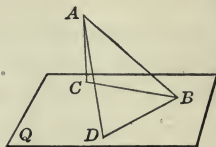**To prove** $\angle ABC < \angle ABD$.

Proof. Make $BD = BC$, and draw AC and AD.

Then $\angle ABC < \angle ABD$.   § 259

586. Definition. The acute angle that a straight line makes with its own projection upon a plane is called the **inclination** of the line to the plane, or the **angle that the line makes with the plane.**

EXERCISES

1. State and prove the converse theorems to § 584.

2. How does the length of the projection of a line upon a plane compare with the length of the line when: (1) the line is parallel to the plane; (2) the line is perpendicular to the plane; (3) the line is neither parallel nor perpendicular to the plane?

3. Which is the greatest angle that an oblique line to a plane makes with any line in the plane and through its foot?

4. Parallel lines make equal angles with a plane.

5. If a straight line intersects two parallel planes it makes equal angles with the parallel planes.

6. Find the projection of a line 16 in. long upon a plane if the angle it makes with the plane is 45°. If 30°. If 60°.

7. What is the locus of a point in a given plane and equidistant from two given points not in the given plane?

8. A rectangle 8 in. by 12 in. is intersected along one of its diagonals by a plane. If the other diagonal makes an angle of 45° with the plane, find the distance from the extremities of this diagonal to the plane.

Ans. 5.098 in.

9. The projection of a given line on each of two planes is a in length. Are the two planes necessarily parallel? Illustrate.

DIEDRAL ANGLES

587. Definitions. An angle formed by two intersecting planes is called a **diedral angle.** The planes are called the **faces** of the diedral angle, and their intersection is called the **edge.**

If a plane Q revolves about an axis AB, and if a line $CD \perp AB$ is taken in Q, then the plane Q generates a diedral angle, and the line CD generates a plane angle since it revolves in a plane. When the plane angle is acute, right, obtuse, etc., the diedral angle generated in connection with it is acute, right, obtuse, etc. That is, the diedral angle and the plane angle are each formed by the same amount of turning.
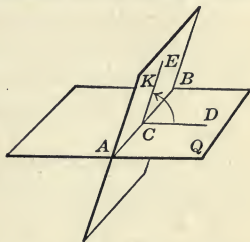
588. Plane Angle. The plane angle generated by CD is called the **plane angle** of the diedral angle. It is formed by two lines, one in each face of the diedral angle, and perpendicular to the edge at the same point.

A diedral angle is read by naming its faces, as the diedral angle QP; or it may be read C-AB-D; or simply AB.

The words *complementary, supplementary, adjacent, vertical,* etc., when applied to diedral angles, have meanings corresponding to their meanings when applied to plane angles.

589. The following facts are readily deduced from the definitions:
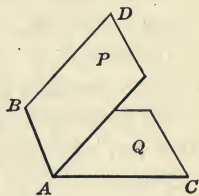
(1) *All plane angles of a diedral angle are equal.*

(2) *If two diedral angles are equal, their plane angles are equal; and conversely.*

(3) *Two diedral angles are proportional to their plane angles; that is, the plane angle of a diedral angle can be taken as the measure of the diedral angle.*

EXERCISES

1. By comparison with the corresponding terms in plane geometry, form definitions applying to diedral angles for right, oblique, vertical, adjacent, supplementary, alternate interior. How many of these can be illustrated by the leaves of an open book?

2. With reference to planes, what corresponds to the following axiom in plane geometry: Through a given point only one straight line can be drawn parallel to another straight line?

3. If one plane intersects another plane, the vertical diedral angles are equal.

4. If the sum of two adjacent diedral angles is two right diedral angles, the exterior faces lie in the same plane.

5. If two planes are cut by a third plane so as to make the alternate diedral angles equal, the two planes are parallel.

6. Take a piece of heavy paper, rule it as shown in figure (1), and cut along *AO*. Fold as shown in (2) and show how it may be used in measuring diedral angles.
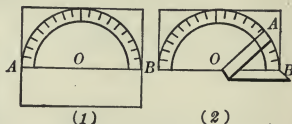
590. Perpendicular Planes. If two planes form a right diedral angle, the two planes are said to be **perpendicular** to each other.

591. Theorem. *If a line is perpendicular to a plane, every plane containing this line is perpendicular to the given plane.*

Given *CD* perpendicular to plane *PQ*, and plane *MN* through *CD* intersecting *PQ* in *AB*.

To prove that plane *MN* is perpendicular to plane *PQ*.

Proof. *CD* ⊥ *AB*.   Why?

In plane *PQ* through *D*, draw *DE* ⊥ *AB*.

Then ∠*CDE* is the plane angle of the diedral ∠*M-AB-Q*.   § 588

But ∠*CDE* is a rt. ∠.   Why?

Hence diedral ∠*M-AB-Q* is a rt. diedral ∠.   § 589 (3)

∴ plane *MN* is perpendicular to plane *PQ*.   Why?
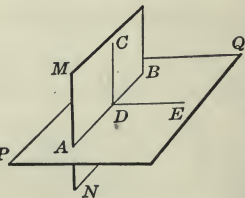
592. Theorem. *If a straight line is parallel to a plane, any plane perpendicular to the line is perpendicular to the plane.*
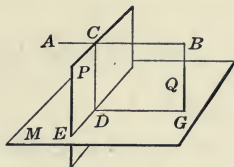
Given $AB \parallel$ plane M, and plane $P \perp AB$ at C.

To prove $P \perp M$.

Proof. From any point B in AB draw $BG \perp M$.   § 574

AB and BG determine a plane Q which intersects plane M in DG, and plane P in CD.

Further	$DG \parallel AB$, and $CD \perp CB.$	Why?
Then	$CD \parallel BG.$	Why?
Therefore	$CD \perp M.$	§ 580
	$\therefore P \perp M.$	§ 591

EXERCISES

1. Through a given point pass a plane perpendicular to a given plane. How many such planes can be drawn?

2. If one line is perpendicular to another line, is every plane containing the first line perpendicular to the second? Prove.

3. A plane determined by a line and its projection upon a given plane is perpendicular to the given plane.

4. If three concurrent lines are each perpendicular to the other two, what can be proved concerning the planes determined by the lines? Illustrate by the walls of a room.

5. If a plane is perpendicular to a line in another plane it is perpendicular to that other plane.

6. State the converse of the theorem of § 592. Is it a true theorem? Prove.

7. A circle is divided into eight equal parts by diameters. A plane containing one of these diameters makes an angle of 45° with the plane of the circle. Find the distance of each division point of the circle from the plane.

593. Theorem. *If two planes are perpendicular to each other a line drawn in one of them perpendicular to their intersection is perpendicular to the other.*

Given plane $MN \perp$ plane PQ, AB their intersection, and $CD \perp AB$ in plane MN.

To prove $CD \perp$ plane PQ.

Proof. Through D and in plane PQ draw $DE \perp AB$.

Then $\angle CDE$ is the plane angle of the right diedral $\angle M\text{-}AB\text{-}Q$.
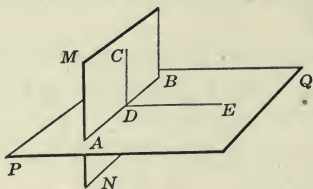
Hence $\angle CDE$ is a rt. $\angle$, and $CD \perp DE$.	Why?
But $\qquad\qquad CD \perp AB$.	Given
$\therefore CD \perp$ plane PQ.	§ 571

594. Theorem. *If two planes are perpendicular to each other a perpendicular to one of them at any point of their intersection will lie in the other.*

Given plane $MN \perp$ plane PQ, and intersecting it in AB, also $CD \perp$ plane PQ at any point D in AB.

To prove that CD must lie in plane MN.

Suggestion. In plane MN draw a line $\perp AB$ at D. Then this line is $\perp$ plane PQ, and must coincide with CD. Why?

595. Theorem. *If two planes are perpendicular to each other a perpendicular to one from any point in the other will lie in the other.*

Given plane $MN \perp$ plane PQ, and intersecting it in AB, also $CD \perp$ plane PQ from any point C in plane MN.

To prove that CD must lie in plane MN.

Suggestion. In plane MN draw a line $\perp AB$ from C. Then this line is $\perp$ plane PQ, and must coincide with CD. Why?

596. **Theorem.** *If each of two intersecting planes is perpendicular to a third plane their line of intersection is perpendicular to the third plane.*

Given planes P and Q, intersecting in AB, and each $\perp$ plane M.

To prove $AB \perp M$.

Proof. Either $AB \perp M$ or it is not.

If AB is not $\perp M$, draw a line $HB \perp M$ at B.

Then HB lies in both P and Q.   § 594

Hence HB is the intersection of P and Q.   Why?

But AB is the intersection of P and Q.   Given

Therefore AB and HB coincide.   § 551

$$\therefore AB \perp M.$$
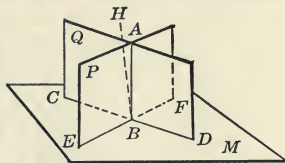
597. **Theorem.** *Through any straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.*

Given plane P and AB a line not $\perp P$.

To prove that one plane $\perp P$ can be passed through AB, and only one.

Proof. From any point A in AB, draw $AC \perp P$.

AB and AC determine a plane $\perp P$.   § 591

$\therefore$ one plane $\perp P$ can be passed through AB.

If another plane $\perp P$ could be drawn through AB, then AB would be $\perp P$ by § 596.

But AB is not $\perp P$.   Given

$\therefore$ only one plane $\perp P$ can be passed through AB.
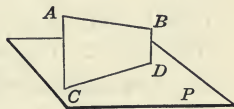
EXERCISES

1. Discuss the theorem of § 597 for the case where AB lies in plane P.

2. If a plane is perpendicular to each of two intersecting planes, it is perpendicular to their intersection.

3. Any point in a plane containing the bisector of an angle, and perpendicular to the plane of the angle, is equally distant from the sides of the angle.

598. Theorem. *Between two straight lines not in the same plane one common perpendicular, and only one, can be drawn.*
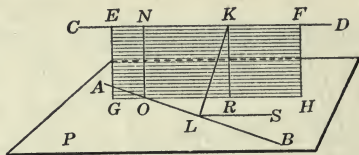
Given *AB* and *CD* two non-coplanar straight lines.

To prove that one common perpendicular to *AB* and *CD*, and only one, can be drawn.

Proof.   Through *AB* pass plane *P* ∥ *CD*.   § 559
Through *CD* pass plane *EH* ⊥ *P*, and intersecting *P* in *GH*.
                                                                    § 597
Then   *GH* ∥ *CD*.   Why?
And   *GH* intersects *AB* in some point *O*.   Why?
In plane *EH* draw *NO* ⊥ *GH* at *O*.
Then *NO* ⊥ *CD* and *NO* ⊥ *AB*.   Why?
∴ one common perpendicular to *AB* and *CD* can be drawn.
Suppose that there is another common perpendicular *KL*.
In plane *P* draw *LS* ∥ *GH*, and in plane *EH* draw *KR* ⊥ *GH*.
Then   *LS* ∥ *CD*, and hence *KL* ⊥ *LS*.   Why?
Therefore   *KL* ⊥ plane *P*.   Why?
But this is impossible for *KR* ⊥ plane *P*.   Why?
Hence   *KL* is not ⊥ *AB*.
∴ only one common perpendicular to *AB* and *CD* can be drawn.

599. Theorem. *The common perpendicular between two non-coplanar lines is the shortest line that can be drawn between them.*

Prove that *NO* is less than any other line *KL*, for *NO* = *KR* and *KR* < *KL*.

600. Definition. The length of the shortest line between two lines is called the **distance** between them.

Compare with §§ 187-190, 577.

601. Theorem. *Every point in a plane that bisects a diedral angle is equidistant from the faces of the angle.*
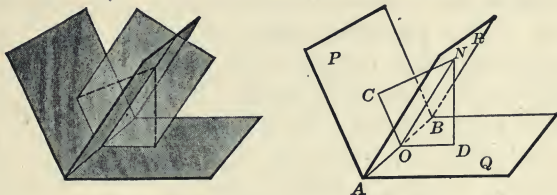
Given the plane R bisecting the diedral angle formed by the planes P and Q, and N any point in plane R. Also $NC \perp P$ and $ND \perp Q$.

To prove $NC = ND$.

Proof. NC and ND determine plane CD.   Why?

Plane CD intersects P in CO, Q in DO, and R in NO.

Plane $CD \perp P$ and also $\perp Q$.   § 591

Then plane $CD \perp AB$.   Why?

And NO, CO, and DO are each $\perp AB$.   Why?

Therefore $\angle NOC$ is the plane angle of diedral $\angle RP$, and $\angle NOD$ is the plane angle of diedral $\angle RQ$.

Hence $\angle NOC = \angle NOD$.   Why?

Show $\triangle NOC \cong \triangle NOD$.

                         $\therefore NC = ND$.   Why?

602. Theorem. *Every point equidistant from the two faces of a diedral angle lies in the plane bisecting the angle.*

EXERCISES

1. Are both of the preceding theorems included in the following? The locus of a point equidistant from the faces of a diedral angle is the plane bisecting the angle. Explain.

2. From any point within a diedral angle perpendiculars are drawn to the faces. Prove that the angle formed by these perpendiculars is supplementary to the plane angle of the diedral angle.

3. The plane angle of a diedral angle is 120°. A point in the bisector of the diedral angle is 16 in. from the edge of the angle. Find the distance of this point from the faces of the angle.

POLYEDRAL ANGLES

603. Definitions. When three or more planes meet at a point they form a **polyedral angle,** or a **solid angle.**

The polyedral angle is formed by a portion of the planes as shown in the figure. The point V in which all the planes meet is the **vertex** of the polyedral angle; the lines, AV, BV, and CV, of intersection of consecutive planes are the **edges;** the portions of the planes lying between the edges are the **faces;** and the angles in the faces between the edges are the **face angles.**
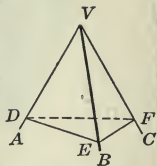
A polyedral angle is read by the letter at the vertex, or by this letter together with a letter on each edge.

Thus, the polyedral angle in the figure is read "polyedral angle V, or V-ABC."

604. Parts of a Polyedral Angle. The face angles are AVB, BVC, and CVA. The diedral angles formed by the faces are the diedral angles of the polyedral angle. The **parts** of a polyedral angle are its face angles and its diedral angles.

For convenience in representing a polyedral angle, a plane is often passed through it cutting all its edges. It should be noted that this plane is not a part of the polyedral angle.

605. Convex and Concave Polyedral Angles. If a plane cuts all the edges of a polyedral angle, but not through the vertex, the intersections of the faces with this plane form a polygon. If this polygon is convex (§ 169), the polyedral angle is said to be **convex.** If the polygon is concave, the polyedral angle is said to be **concave.**
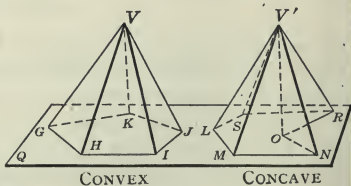
In this text only convex polyedral angles will be considered, unless otherwise stated.

606. Classification. A polyedral angle that has three faces is called a **triedral angle.** The words **tetraedral, pentaedral, hexaedral,** etc., may be applied when the polyedral angle has four, five, six, etc., faces.

607. Congruent Polyedral Angles. If the corresponding parts of two polyedral angles are equal and arranged in the same order, the polyedral angles are **congruent.**

Thus, in the figure, the face angles are equal and arranged in the same order, that is, $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, and $\angle CVA = \angle C'V'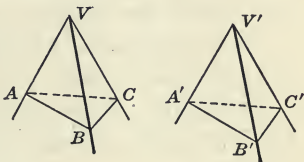A'$. Also in the diedral angles, $\angle AV = \angle A'V'$, $\angle BV = \angle B'V'$, and $\angle CV = \angle C'V'$, the arrangement is in the same order.

If the corresponding parts of two polyedral angles are equal and arranged in the opposite order, the polyedral angles are said to be **symmetric.** These will be considered later (§ 836).

EXERCISES

1. Does the size of a polyedral angle depend upon the lengths of its edges? How do the number of edges, the number of diedral angles, and the number of face angles of a polyedral angle compare?

2. Construct from stiff paper a triedral angle having face angles equal to 50°, 70°, and 90°. Can you tell the number of degrees in the diedral angles?

The paper may be cut as indicated in the figure, folded along the dotted lines and pasted.

3. How many degrees in each of the face angles and the diedral angles of the polyedral angle formed by the walls of a room?

4. The face angles and diedral angles of a convex polyedral angle are each less than 180°.

5. Bearing the plane geometry definitions in mind, define vertical polyedral angles. Are they congruent or symmetric? Explain how, if it is possible to have two vertical triedral angles congruent.

6. Could a triedral angle have one right diedral angle? Could it have two? Three? Give illustrations.
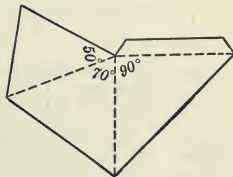
608. Theorem. *The sum of two face angles of a triedral angle is greater than the third face angle.*

Given the triedral $\angle V\text{-}ABC$.

To prove that the sum of two face angles is greater than the third face angle.

Proof. Three cases arise: (1) all the face angles equal; (2) two face angles equal; (3) no two face angles equal.

Cases (1) and (2) are left to the student.

In case (3) suppose $\angle AVB$ is the greatest.

In the face AVB, construct $\angle AVD = \angle AVC$, take $VD = VC$, and draw AB, AC, and BC.

Then $\qquad\qquad\qquad \triangle AVC \cong \triangle AVD.$ $\qquad\qquad$ § 247

And $\qquad\qquad\qquad\quad AC = AD.$ $\qquad\qquad\qquad$ Why?

$\qquad\qquad\qquad AC + CB > AD + DB.$ $\qquad\qquad$ § 182

Hence $\qquad\qquad\qquad CB > DB.$ $\qquad\qquad\qquad$ § 174

In $\triangle CVB$ and DVB, $BV = BV$, and $CV = DV$, but $CB > DB$.

Therefore $\qquad\qquad \angle CVB > \angle DVB.$ $\qquad\qquad$ § 259

Then $\qquad \angle AVC + \angle CVB > \angle AVD + \angle DVB.$ $\qquad$ § 173

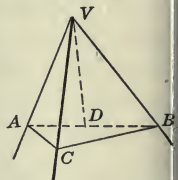$\qquad\qquad \therefore \ \angle AVC + \angle CVB > \angle AVB.$

Or the sum of two face angles is greater than the third.

EXERCISES

1. Any face angle of a triedral angle is greater than the difference of the other two.

2. State the theorems in plane geometry that correspond to the theorems of § 608 and Exercise 1.

There is a close analogy between the plane triangle and the triedral angle. Many theorems concerning plane triangles can be changed to theorems concerning triedral angles by replacing the word *side* by *face angle,* and the word *angle* by *diedral angle.*

3. State theorems for triedral angles analogous to the theorems of plane geometry that have to do with congruent triangles.

4. Referring to the figure of § 608, $\angle AVC = 55°$ and $\angle CVB = 65°$. Make a statement regarding the number of degrees in $\angle AVB$.

609. Theorem. *The sum of the face angles of any convex polyedral angle is less than four right angles.*
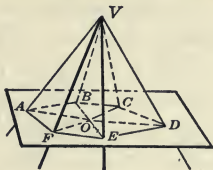
Given the convex polyedral $\angle V\text{-}ABCDEF$.

To prove $\angle AVB + \angle BVC + \cdots \angle FVA < 4$ rt.$\angle$.

Proof. Let a plane intersect the edges of the polyedral angle in A, B, C, etc., and intersect the faces in AB, BC, CD, etc. Join any point in the polygon thus formed to the vertices of the polygon.

Then $\qquad \angle VBA + \angle VBC > \angle ABC$. $\qquad$ § 608

Similarly $\qquad \angle VCB + \angle VCD > \angle BCD$, etc.

Add these and the result is that the sum of all the base angles of the triangles with vertices at V is greater than the sum of all the base angles of the triangles with vertices at O.

Then, since the sum of all the angles of the triangles with vertices at V is equal to the sum of those with vertices at O, the sum of the angles about V is less than the sum of the angles about O. $\qquad$ § 178

But the sum of the angles about $O = 4$ rt.$\angle$. $\qquad$ Why?

$$\therefore \angle AVB + \angle BVC + \cdots \angle FVA < 4 \text{ rt.}\angle.$$

EXERCISES

1. Can a polyedral angle have for its face angles three angles from an equilateral triangle? Can it have four? Five? Six? Why?

2. Can a polyedral angle have as face angles the angles of a square? Of a regular pentagon? Of a regular hexagon? Why?

3. Any face angle of a polyedral angle is less than the sum of the others.

4. The sum of the face angles of a polyedral angle is 300°. What is the greatest value that any one of the face angles may have?

5. One of the face angles of a triedral angle is 60°. What are the limitations on the size of each of the other face angles?

610. Theorem. *Two triedral angles are congruent if the three face angles of one are equal respectively to the three face angles of the other, and arranged in the same order.*

Given triedral angles V and V', having $\angle AVB = \angle A'V'B'$, $\angle BVC = \angle B'V'C'$, $\angle CVA = \angle C'V'A'$, and arranged in the same order.
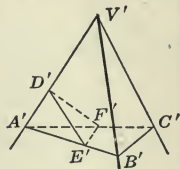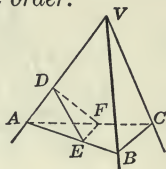
To prove triedral $\angle V \cong$ triedral $\angle V'$.

Proof. On the edges of triedral angles V and V' lay off the six equal segments VA, VB, VC, $V'A'$, $V'B'$, and $V'C'$, and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then there are three pairs of congruent isosceles triangles as follows: $\triangle AVB \cong \triangle A'V'B'$, $\triangle BVC \cong \triangle B'V'C'$,

$$\triangle CVA \cong \triangle C'V'A'. \qquad \text{Why?}$$

Hence $AB = A'B'$, $BC = B'C'$, and $CA = C'A'$. Why?

And $\triangle ABC \cong \triangle A'B'C'$. Why?

Also $\measuredangle\ VAC$ and VAB are acute. Why?

From any point D in VA construct DE in face AVB and DF in face AVC, each $\perp VA$. These lines meet AB and AC respectively in E and F. Why?

Draw EF.

Take $V'D' = VD$ and in a similar manner construct, $D'E'$, $D'F'$, and $E'F'$.

Then, since $AD = A'D'$ and $\angle DAE = \angle D'A'E'$,

$$\triangle ADE \cong \triangle A'D'E' \qquad \S\ 97$$

Hence $AE = A'E'$ and $DE = D'E'$. Why?

Similarly $AF = A'F'$ and $DF = D'F'$.

Therefore $\triangle AEF \cong \triangle A'E'F'$. § 247

And $EF = E'F'$. Why?

Therefore $\triangle DEF \cong \triangle D'E'F'$. § 113

And $\angle EDF = \angle E'D'F'$. Why?

Hence diedral $\angle VA =$ diedral $\angle V'A'$. § 589 (3)

Similarly, prove the other two pairs of diedral angles equal.

$$\therefore \text{ triedral } \angle V \cong \text{ triedral } \angle V'. \qquad \S\ 607$$

LOCI

611. As has been assumed on the previous pages, the idea of a locus here is the same as in plane geometry.

Here, too, the proof of a locus theorem must, as in plane geometry, consist of two parts:

(1) *That all points in the figure satisfy the given conditions.*

(2) *That all points that satisfy the given conditions are in the figure.*

In the place of (2) one may prove that all points not in the figure do not satisfy the given conditions.

612. In plane geometry it was noticed that usually a locus satisfying one condition is a line or group of lines, and a locus satisfying two conditions is a point or group of points. There, however, a further condition is assumed, namely, that the locus is confined to a plane.

In solid geometry usually, one condition will confine the locus to a surface or group of surfaces, two conditions will confine the locus to a line or group of lines, and three conditions will confine the locus to a point or group of points.

Here, as in plane geometry, one of the greatest benefits derived from the study of loci is through the imaging and constructing figures, rather than through the proofs of the loci theorems. This, however, does not mean that the proofs should be neglected.

EXERCISES

1. What is the locus of points equally distant from two given points?

2. What is the locus of points equally distant from three given points?

3. What is the locus of points equally distant from four given points?

4. What is the locus of points equally distant from two intersecting lines and in a given plane?

5. What is the locus of the end point of a line-segment of fixed length, that moves so as to remain parallel to a given line and have one end in a given plane?

6. What is the locus of points in a given plane equally distant from three given points not in a line?

7. What is the locus of points equally distant from two given parallel lines?

8. What is the locus of the middle point of a line-segment of fixed length, that moves so as to have its end points in two parallel planes?

9. What is the locus of a point equally distant from two parallel planes and equally distant from the faces of a diedral angle?

10. What is the locus of all lines that pass through a given point and are parallel to a given plane not containing the point?

11. What is the locus of the points within a diedral angle and equally distant from its two faces?

12. What is the locus of all points that are twice as far from a line in a plane as from the plane?

13. What is the locus of the points within a triedral angle and equally distant from its three faces?

14. What is the locus of the points equally distant from the edges of a triedral angle?

15. What is the locus of a line making equal angles with each of two intersecting lines? What other exercise in this list has the same locus as this?

16. What is the locus of points in one of two given non-coplanar lines equally distant from two given points in the other?

17. What is the locus of a point that moves so that its distance from one of two given parallel planes is to its distance from the other as 1 : 3?

18. What is the locus of a point whose distances from the faces of a diedral angle are respectively 5 in. and 8 in.?

19. What is the locus of the projections of a given point upon the planes containing a given line?

Many questions concerning loci cannot be discussed in elementary geometry as they lead to forms not there considered, for in elementary geometry only a limited number of forms such as lines, circles, planes, cones, cylinders, and spheres are considered. The question: What is the locus of the middle points of all transversals of two non-coplanar lines? leads to none of these.

In general, the locus of points equally distant from two different elements, as a point and a line, a line and a plane, or two non-coplanar lines cannot be considered in elementary geometry.

QUESTIONS

1. State the different relations that two lines may have to each other. That a line may have to a plane. That two planes may have to each other.

2. State the different ways of determining that a straight line lies in a plane.

3. What is the intersection of a line and a plane? Of two planes? Of three planes?

4. State the different ways of determining that a line is parallel to a plane.

5. State the different ways of determining that a line is perpendicular to a plane.

6. State the different ways of determining that two planes are parallel.

7. State the different ways of determining that two planes are perpendicular.

8. How many intersecting lines can be parallel to the same line? To the same plane?

9. How many intersecting planes can be parallel to the same line? To the same plane?

10. Must planes be parallel if they contain parallel lines? Are lines necessarily parallel if they are in parallel planes?

11. How many lines through a point can be perpendicular to the same line? To the same plane?

12. How many intersecting planes can be perpendicular to the same line? To the same plane?

13. What are the relations of the following to each other: (1) lines parallel to the same line; (2) planes parallel to the same plane; (3) lines parallel to the same plane; (4) planes parallel to the same line; (5) lines perpendicular to the same line; (6) lines perpendicular to the same plane; (7) planes perpendicular to the same line; (8) planes perpendicular to the same plane?

14. If two planes are perpendicular to each other, what lines in one are perpendicular to the other?

15. Can the projection of a straight line upon a plane be a curve? Can the projection of a circle be a straight line? Explain.

16. Can the projection of a square be a square? A rectangle? A parallelogram? Explain.

GENERAL EXERCISES

COMPUTATION

1. Two fixed points, A and B, are 10 in. apart. A point P moves so as to keep 8 in. from A and B. Find the length of the locus of P.

2. Two fixed points, A and B, are 12 in. apart. A point P moves so as to keep 10 in. from A and 4 in. from B. Find the radius of the circle generated.   *Ans.* 2.5 in.

3. A 20-foot pole is placed obliquely in a river. One end of the pole is on the bottom of the river and the other end is 3 ft. above the surface. Find the depth of the river if the length of the part of the pole under water is 15 ft.

4. Plane Q is perpendicular to plane P. Find the shortest line AED that can be drawn from a point A in Q to a point D in P if $AB = 5$ ft., $CD = 10$ ft., and $BC = 20$ ft.
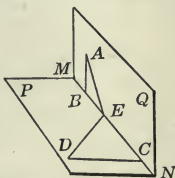
SUGGESTION. Consider the figure when Q is turned into the same plane as P.

5. The room shown in the figure has a length of 40 ft., a width of 20 ft., and a height of 12 ft. Find the length of the shortest line in the walls of the room from B to H.
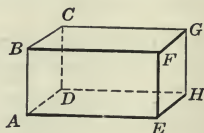
6. The sum of the face angles of a trihedral angle is 148°. What is the greatest value a face angle can have?

7. In the figure of a hammock support with the dimensions as given, find the length of AB if $AB = CB = DE = FE$.
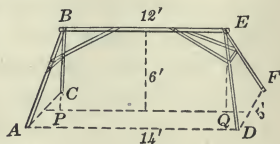
8. The roof of a house makes an angle of 30° with the plane of the plates. The roof is 18 ft. by 30 ft. Find the area of its projection upon the plane of the plates.
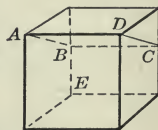
9. In the figure of a cube an edge is 18 in. and BE is 10 in. Find the area of $ABCD$.

10. Having given the dimensions as shown in the figure, find the area of section S to two decimal places. All the face planes that meet are perpendicular to each other.
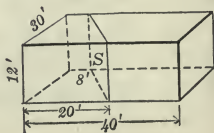
11. The figure shows the plan for a half pitch roof. The dimensions are as given and the rafters are to be placed 1 ft. 6 in. from center to center. Find the lengths of the hip rafters, common rafters, and jack rafters. *AB* is a hip rafter, *CD* is a common rafter, and *EF* is a jack rafter.

THEOREMS AND PROBLEMS
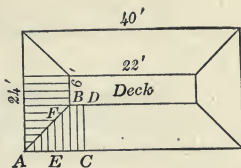
1. If a series of parallel planes intercept equal segments on one straight line, they intercept equal segments on any other straight line that they intersect.

2. If a straight line and a plane are each perpendicular to the same straight line, they are parallel.

3. If two planes are perpendicular to each other, a line perpendicular to one and not in the other is parallel to the other.

4. Prove that the sides of an isosceles triangle make equal angles with any plane in which the base lies.

5. From a point within a triedral angle perpendiculars are drawn to the three faces. These perpendiculars are the edges of a second triedral angle. Prove that the face angles of one are supplementary to the diedral angles of the other.

6. Three planes, *M*, *N*, and *Q*, each perpendicular to the other two pass through a common point. Show that a point 5 in. from *M*, 8 in. from *N*, and 10 in. from *Q* may be in any one of eight positions.

7. Given two non-coplanar lines, to construct a plane upon which the projections of the two lines will be parallel.

8. A ray of light from the source *L* is reflected by the mirror *M* and enters the eye at *E*. The path travelled by the light is the shortest possible path from *L* to the mirror and then to *E*. Show how to determine where the ray of light strikes the mirror.
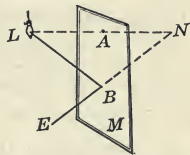
9. Three non-coplanar lines meet at a point. Construct a line that will make equal angles with the three lines.

10. Given a polyedral angle with four faces, to construct a plane that will intersect the four faces so that the intersection will be a parallelogram.

SUGGESTION. Pass a plane parallel to one face and intersecting the opposite face. Determine a line in the first face equal to this intersection and parallel to it. These two lines are parallel and hence determine a parallelogram.

11. Prove that if a line is parallel to one plane and perpendicular to another, the two planes are perpendicular to each other.

12. If a plane be passed through a diagonal of a parallelogram, the perpendiculars to it from the extremities of the other diagonal are equal.

Plane MN passes through diagonal BD of parallelogram $ABCD$. Prove perpendiculars AP and CQ are equal.
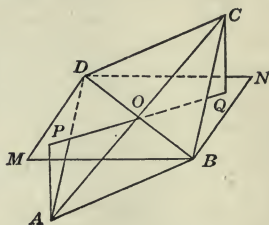
13. Describe how to locate a point that is 4 in. from face of a triedral angle, 5 in. from another face, and 3 in. from the third face.

14. If a series of parallel planes cut all the edges of a triedral angle, the intersections of the planes with the faces form similar triangles.

15. If from P, any point within the diedral angle A-BC-D, PM and PN are drawn perpendicular to the faces BD and AC respectively, and MS is drawn perpendicular to AC, then NS is perpendicular to BC.
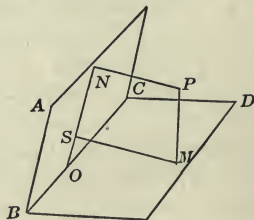
16. The three planes bisecting the diedral angles of a triedral angle meet in a line.

SUGGESTION. Show that two of them meet in a line, and then show that this line lies in the third plane.

Compare this exercise with Exercise 13, page 308.

17. All points within a triedral angle and equally distant from its three faces, lie in the line of intersection of the planes that bisect the diedral angles.
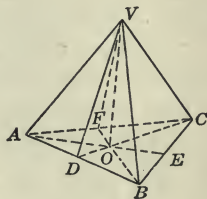
18. The planes through the bisectors of the face angle of a triedral angle and perpendicular to the planes of the respective faces, meet in a line.

CHAPTER VII

POLYEDRONS, PRISMS, CYLINDERS

613. In the present and following chapters will be considered some of the solids most commonly observed in nature, and very frequently used in architecture, engineering, and the arts. Besides finding the areas and volumes of these solids it will be necessary to investigate the relations of their parts, and to study the plane figures formed when the solids are cut by planes.

614. Polyedrons. A **polyedron** is a solid entirely bounded by planes.
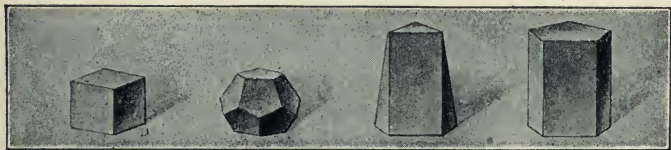
The intersections of the bounding planes are the **edges** of the polyedron. The points in which the edges intersect are the **vertices**. The polygons bounded by the edges are the **faces**. The faces taken together make up the **surface,** and the area of this surface is the **area** of the polyedron. The amount of space enclosed by the surface is the **volume** of the polyedron.

615. Sections. The plane figure formed on a plane passing through a solid, and bounded by the intersections of the plane with the surface of the solid is called a **section of the solid.** It is evident that the section of a solid is a **closed figure** (§ 146), and that the section of a polyedron is a *polygon.*
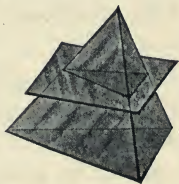
616. Convex Polyedrons. A polyedron is **convex** if every section of it is a convex polygon. Otherwise it is **concave.**

Only convex polyedrons are considered unless otherwise stated.

617. Prismatic and Cylindrical Surfaces. A moving straight line that always remains parallel to its original position, and intersects a fixed straight line, generates a **plane.** Why?
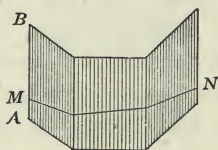
PLANE   PRISMATIC SURFACE   CYLINDRICAL SURFACE

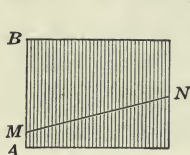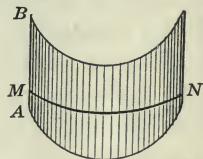A moving straight line that always remains parallel to its original position, and intersects a broken line not coplanar with it, generates a **prismatic surface.**

A moving straight line that always remains parallel to its original position, and intersects a plane curved line not coplanar with it, generates a **cylindrical surface.**

AB is the original position of the moving line, and MN is the fixed line.

618. The moving line is called the **generatrix,** and the fixed line or curve the **directrix.**

The generatrix in any position is called an **element** of the surface generated.
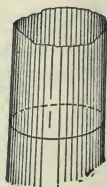
619. Closed Surface. If the directrix is a closed line, the prismatic or cylindrical surface is **closed.**

620. Sections of Prismatic and Cylindrical Surfaces. A plane cutting all the elements of a closed prismatic or cylindrical surface cuts the surface in a closed line. The figure bounded by this closed line is a **section.** If the cutting plane is perpendicular to an element, the section is a **right section;** if not, it is an **oblique section.** Two sections made by parallel planes are **parallel sections.**
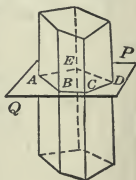
In the figure, $ABCDE$ is a right section.

PRISMS AND CYLINDERS

621. A **prism** is the solid formed by a closed prismatic surface and two parallel cutting planes.

622. A **cylinder** is the solid formed by a closed cylindrical surface and two parallel cutting planes.

623. The sections formed by the cutting planes are called the **bases** of the prism or cylinder. It follows from § 620 that the bases of a prism are polygons.
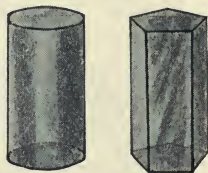
624. The polygons formed on the prismatic surface between the bases are called the **lateral faces.** Intersections of the lateral faces are **lateral edges,** and intersections of the lateral faces with the bases are **base edges.**

625. The **altitude** of a prism, or cylinder, is the perpendicular distance between its bases.

626. The **lateral area** of a prism, or cylinder, is the prismatic, or cylindrical, surface between the bases. The **total area** is the lateral area together with the areas of the bases.

In the figure read the bases, faces, lateral edges, base edges, altitude.
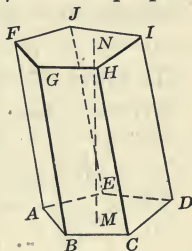
627. The following facts about prisms and cylinders are readily proved by the definitions and previous theorems:

(1) *The lateral edges of a prism, or the elements of a cylinder, are equal.*

(2) *The lateral faces of a prism are parallelograms.*

(3) *The right section of a prism, or cylinder, is perpendicular to all the lateral edges, or elements.*

(4) *The section of a prism made by a plane parallel to a lateral edge is a parallelogram.*

(5) *The section of a cylinder made by a plane containing two elements is a parallelogram.*

628. Theorem. *Parallel sections of a prism are congruent.*

Given the prism MN, and the parallel sections AD and $A'D'$.

To prove polygon $AD \cong$ polygon $A'D'$.

Proof. $AB \parallel A'B'$. Why?

Hence $AB = A'B'$. § 158

Similarly $BC = B'C'$, $CD = C'D'$,
$DE = D'E'$, $EA = E'A'$.

Also $\angle EAB = E'A'B'$, $\angle ABC = \angle A'B'C'$,
etc. § 567

The two polygons AD and $A'D'$ are then mutually equilateral and mutually equiangular. They then can be proved congruent by superposition.

$\therefore$ polygon $AD \cong$ polygon $A'D'$.

629. Theorem. *Parallel sections of a cylinder are congruent.*

Given the cylinder MN, and the parallel sections P and Q made by the planes.

To prove $P \cong Q$.

Proof. Take two points A and B on the perimeter of P, and let C be any third point of that perimeter.

Draw the elements through A, B, and C to the points A', B', and C' respectively in the perimeter of $Q;$ and draw AB, BC, CA, $A'B'$, $B'C'$, and $C'A'$.

Then $\triangle ABC \cong \triangle A'B'C'$. Why?

And, if P is superposed on Q with AB coinciding with $A'B'$, C will fall upon C'. Why?

But C is any point in the perimeter of P.

Hence every point in the perimeter of P will fall upon a point of Q, and conversely.

$\therefore P \cong Q.$

630. Theorem. *Every section of a prism, or cylinder, parallel to the bases is congruent to them.*
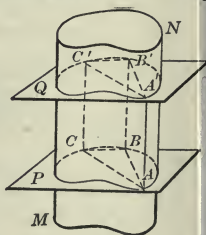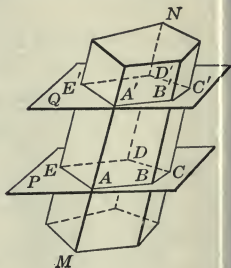
631. A **right prism,** or **cylinder,** is one whose bases are right sections.

632. An **oblique prism,** or **cylinder,** is one whose bases are oblique sections.

633. A prism is **triangular, quadrangular, pentagonal,** etc., according as its bases have *three, four, five,* etc., sides.

634. A **regular prism** is a right prism whose bases are regular polygons.

635. A **circular cylinder** is a cylinder whose bases are enclosed by circles. If the bases are also right sections, the cylinder is a **right circular cylinder.**

636. The line joining the centers of the bases of a circular cylinder is called its **axis.** The radius of the base of a circular cylinder is called the **radius** of the cylinder.

637. Theorem. *A lateral edge of a right prism, or an element of a right cylinder, is equal to the altitude.*   §§ 620, 563.

638. Theorem. *The lateral faces of a right prism are rectangles.*   Apply § 627 (2).

639. Theorem. *If a rectangle revolves about one of its sides as an axis, it generates a right circular cylinder; and conversely.*

Show that the side opposite the axis remains parallel to its original position and moves along a curve. Also show that the other two sides generate parallel bases that are circular, and are perpendicular to the elements.
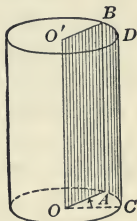
640. Cylinder of Revolution. Since a right circular cylinder may be generated by revolving a rectangle about one of its sides as an axis, it is often called a **cylinder of revolution.**

641. Plane Tangent to a Cylinder. A plane is **tangent** to a cylinder if it touches the cylindrical surface in one element only.

642. Theorem. *If a plane is tangent to a circular cylinder, its intersection with the plane of a base is tangent to that base.*

Given the cylinder OO' and the tangent plane P, meeting the cylinder in the element AB, and intersecting the plane of base O in DE.
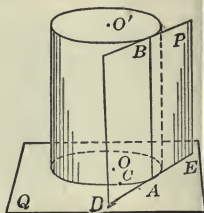
To prove that DE is tangent to base O.

Proof. DE has one point in common with $\odot O$.   Why?

Also DE has only one point in common with $\odot O$, for if it had another point as C in common with $\odot O$, plane P would have a point outside the element AB in common with the cylinder, which is impossible.   Why?

DE also lies in the plane of $\odot O$.   Why?

$\therefore DE$ is tangent to $\odot O$.   § 285

643. Theorem. *The plane determined by a tangent to a base of a circular cylinder and the element through the point of contact is tangent to the cylinder.*

EXERCISES

1. The bases of prisms, or cylinders, are congruent.

2. Would the theorem of § 628 be true if the sections were concave?

3. A line drawn parallel to the elements of a circular cylinder from the center of one base, intersects the other base at its center.

4. The axis of a circular cylinder is parallel to its elements.

5. The axis of a circular cylinder passes through the center of all sections parallel to the bases.

6. Every lateral edge of a prism is parallel to the plane determined by any other two lateral edges.

7. What is the locus of all points equidistant from a given straight line?

8. Show that the plane determined by an element of a circular cylinder and the center of one base will contain the axis.

AREAS OF PRISMS AND CYLINDERS

644. Theorem. *The lateral area of a prism is equal to the product of the perimeter of a right section and a lateral edge.*

Given the prism MP', and its right section $ABCDE$.

To prove $S = pe$, where S denotes lateral area, p the perimeter of a right section, and e a lateral edge.

Proof. Area of $\square MN' = e \cdot AB$. § 355

Similarly area of $\square NO' = e \cdot BC$,
            etc.

Since the lateral area is the sum of the faces, adding these gives:

$$S = e(AB + BC + \cdots EA). \qquad \text{Why?}$$

But $\qquad AB + BC + \cdots EA = p. \qquad$ Why?

$$\therefore S = pe. \qquad \text{Why?}$$
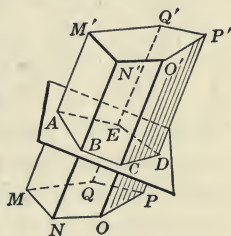
645. Theorem. *The lateral area of a right prism is equal to the product of the perimeter of its base and its altitude.*

That is, $S = ph$, where S denotes lateral area, p perimeter of base, and h altitude.

646. Definitions. An **inscribed prism** is a prism whose bases are respectively inscribed in the bases of a cylinder. The cylinder is then said to be **circumscribed** about the prism.

A **circumscribed prism** is a prism whose bases are respectively circumscribed about the bases of a cylinder. The cylinder is then said to be **inscribed** in the prism.
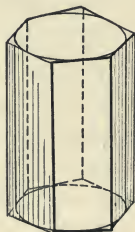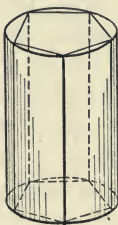
647. It is evident that the lateral edges of an inscribed prism are elements of the cylinder; and that the lateral faces of the circumscribed prism are tangent to the cylinder by § 641.

648. Area of Cylinder. As in § 480, with the circle, the general idea of the measurement of a plane area cannot be used here, for it is impossible to apply a plane unit of area to a cylindrical surface and thus find its numerical measure.

Practically, however, the lateral area of a right circular cylinder can be found by considering the surface as straightened out and measured

$$S = 2\pi r h$$
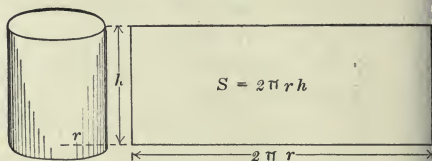
as a rectangle having a length equal to the circumference of the base, and with a width equal to the altitude of the cylinder.

It remains to devise a method for measuring the surface of a cylinder, which is accurate and also geometric.

649. In the circular cylinder of base AB, a prism having a regular polygon for base is inscribed, and the number of sides is doubled indefinitely according to § 463. The area of the cylinder is a constant and that of the prism is a variable. It is evident that the area of each prism is greater than the area of the prism of half the number of faces.
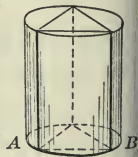
It is also evident that, at all times, the area of the cylinder is greater than the area of each successive prism. By doubling the number of faces of the prism indefinitely, the area continually increases, and can be made to differ from the area of the cylinder by less than any assigned value, that is, *it can be made just as nearly equal to the area of a cylinder as desired.*

Similarly, by circumscribing a prism having a regular polygon for base about a circular cylinder and doubling the number of faces indefinitely, the area continually decreases, and can be made to differ from the area of the cylinder by less than any assigned value, that is, *it can be made just as nearly equal to the area of the cylinder as desired.*
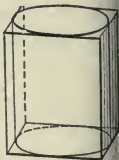
650. Volume. Further, it is evident that the method of the preceding paragraphs can be as readily applied to the volume of a cylinder as to its area. It is thus seen that the volume of an inscribed or circumscribed prism, having a regular polygon for base, as the number of faces is indefinitely doubled, *can be made just as nearly equal to the volume of the cylinder as desired.*

651. From the preceding discussion, the following theorem may be considered established:

Theorem. *If prisms whose bases are regular polygons are inscribed in and circumscribed about a circular cylinder, and if the number of faces of the prisms is indefinitely doubled,*

(1) *The perimeter of a right section of the cylinder is the common limit of the perimeters of right sections of the prisms.*

(2) *The lateral area of the cylinder is the common limit of the lateral areas of the prisms.*

(3) *The volume of the cylinder is the common limit of the volumes of the prisms.*

652. Theorem. *The lateral area of a circular cylinder is equal to the product of the perimeter of a right section and the length of an element.*

Given a circular cylinder.

To prove $S = pe$, where S denotes lateral area, p the perimeter of a right section, and e the length of an element.

Proof. Inscribe a prism, whose base is a regular polygon, in the cylinder, and let S' denote its lateral area and p' the perimeter of a right section.
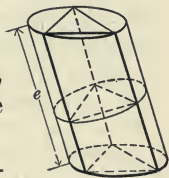
By doubling indefinitely the number of faces of the prism,

$$S' \rightarrow S, \text{ and } p' \rightarrow p. \qquad \S \ 652 \ (1) \text{ and } (2)$$

Then $\qquad\qquad\qquad p'e \rightarrow pe. \qquad\qquad\qquad \S \ 485 \ (2)$

But $\quad S' = p'e$, being variables that are always equal. $\quad \S \ 644$

$$\therefore S = pe, \qquad\qquad\qquad \S \ 485 \ (1)$$

653. Theorem. *The lateral area of a right circular cylinder is equal to the product of its altitude and the circumference of its base.*

That is, $S = ch$, where S denotes lateral area, c circumference of base, and h altitude.

EXERCISES

1. Show that the lateral area and the total area of a right circular cylinder of altitude h and radius r, are given by the formulas:

$S = 2\pi rh$, $T = 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$.

In the following use the notation of Exercise 1.

2. Given $r = 8$ and $h = 10$; find S and T.

3. Given $r = 10$ and $S = 200$; find h and T.

4. Given $h = 16$ and $S = 256$; find r and T.

5. Given $r = 12$ and $T = 4800$; find h and S.

6. Given $h = 25$ and $T = 4500$; find r and S.

7. Given $S = 2000$ and $T = 2200$; find r and h.

8. Solve the formula $T = 2\pi rh + 2\pi r^2$, (1) for r in terms of h and T, (2) for h in terms of r and T.

9. From the formulas of Exercise 1, find r in terms of S and T.

10. A peck measure made of sheet iron has a diameter of 8 in. and a depth of 10.7 in. Find the number of square inches of sheet iron in it.

11. Find the area (no cover) of a wash boiler if the bottom is in the form of a rectangle with a semicircle at each end. The rectangle is 10 in. by 14 in., and the depth of the boiler is 16 in.

12. A steam boiler has a diameter of 72 in., is 18 ft. long and contains 70 tubes each having a diameter of 4 in. extending lengthwise of the boiler. Find the heating surface of the boiler, using "one-half" in the rule below.   *Ans.* 1505 sq. ft.
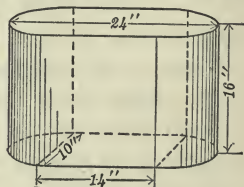
RULE. In finding the heating surface of a horizontal boiler, it is customary to take one-half to two-thirds the lateral area of the shell, the lateral area of the tubes, one-half to two-thirds the area of the ends of the boiler, and subtract the area of both ends of the tubes.

13. Find the heating surface of a horizontal tubular boiler 12 ft. long, 5 ft. in diameter, and having 52 tubes $2\frac{1}{2}$ in. in diameter. Use "two-thirds" in the preceding rule.   *Ans.* 556.7 sq. ft.

14. The base of a right prism is a triangle whose sides are 12 ft., 15 ft., and 17 ft., and its altitude is $8\frac{1}{2}$ ft. Find its lateral area and total area.

15. Find the lateral area of a regular hexagonal prism each side of whose base is 4 in. and whose altitude is 16 in. Find the total area.

16. Derive the formula $S = 0.2618dl$, where S is the area of the surface of a cylindrical pipe in square feet, d the diameter in inches, and l the length in feet.

17. Six lines of steam pipes of diameter 2 in. extend along one side and an end of a room 40 ft. by 30 ft. Find the number of square feet of heating surface.

18. Small metal boxes used for various purposes are made from sheet metal as follows: Blanks of the proper shape are first cut from the sheet metal. These are then pressed into the required form in a die. In computing the size of the

Blank *Box*

blank, it is assumed that it has the same area as the finished article.
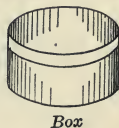
Find the diameter of the blank for the cover of a tin pail for holding lard, if the cover has a diameter of $6\frac{1}{2}$ in. and the flange is $\frac{5}{8}$ in.

19. A cylindrical metal box for holding paper fasteners is $1\frac{3}{4}$ in. in diameter and $1\frac{1}{2}$ in. deep, and the cover has a $\frac{1}{2}$-inch flange. Find the diameter of the blanks for box and cover. *Ans.* 3.68 in.; 2.56 in.

20. What is the locus of points 12 in. distant from two parallel lines 18 in. apart?

21. Holes for rivets are often punched in metal plates. The pressure required for such work, when the material is wrought iron, is 55,000 pounds per square inch of the cut surface in. For example, a hole having a circumference of 2 in. punched in a $\frac{1}{4}$-in. plate would require a pressure of $2 \times \frac{1}{4} \times 55,000$ lb., that is, the area of the cylindrical surface sheared off times 55,000 lb.

Find the pressure required to punch a $\frac{1}{4}$-in. round hole through a piece of sheet iron $\frac{1}{8}$-in. thick.

22. Find the pressure necessary to punch a $\frac{1}{2}$-inch round hole through a boiler plate $\frac{5}{8}$ in. thick if 60,000 pounds pressure is required per square inch of surface cut.

23. A rectangular box is to be made of sheet steel $\frac{1}{32}$ in. thick. Find the number of pounds the punch press must strike in order to cut out the blank if it is rectangular $5\frac{3}{4}$ in. by $2\frac{3}{4}$ in., has squares $\frac{7}{8}$ in. on a side cut from each corner, and has two holes $\frac{3}{16}$ in. in diameter punched in the bottom. Use 60,000 pounds per square inch. *Ans.* 30,297 lb.

CONGRUENT AND EQUIVALENT SOLIDS

654. Congruent Solids. Two solids are said to be **congruent** if they can be made to coincide completely in all their parts.

655. Equivalent solids are those having the same volume.

656. Theorem. *Two prisms are congruent if three faces including a triedral angle of one are congruent respectively to three faces including a triedral angle of the other, and are arranged in the same order.*
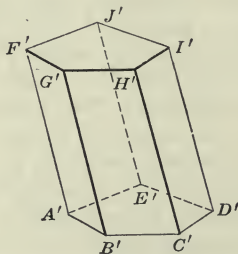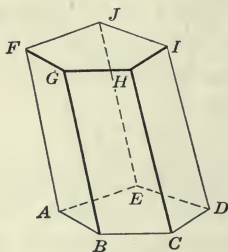
Given prism AI and $A'I'$, with face $AJ \cong$ face $A'J'$, face $AG \cong$ face $A'G'$, and face $AD \cong$ face $A'D'$.

To prove prism $AI \cong A'I'$.

Proof. $\angle BAF = \angle B'A'F'$, $\angle FAE = \angle F'A'E'$,

and $\angle BAE = \angle B'A'E'$.   Why?

Then triedral $\angle A \cong$ triedral $\angle A'$.   § 610

Place the prism $A'I'$ upon prism AI so that triedral $\angle A'$ shall coincide with its congruent triedral $\angle A$.

Then face $A'J'$ coincides with its congruent face AJ; $A'G'$ with its congruent face AG; and $A'D'$ with its congruent face AD.

And points F', G', and J' fall upon F, G, and J respectively.

Further $C'H'$ coincides with CH.   Why?

Hence H' falls upon H.

Similarly I' falls upon I.

            $\therefore$ prism $AI \cong$ prism $A'I'$.   § 654

657. Theorem. *Two right prisms, or two right cylinders, are congruent if they have congruent bases and equal altitudes.*

658. Truncated Prism. A portion of a prism included between the base and a section oblique to the base is called a **truncated prism.**

659. Theorem. *Two truncated prisms are congruent if three faces including a triedral angle of one are congruent respectively to the three faces including a triedral angle of the other, and are arranged in the same order.*
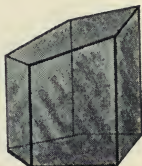
660. Theorem. *An oblique prism is equivalent to a right prism whose base is a right section of the oblique prism, and whose altitude is a lateral edge of the oblique prism.*

Given the oblique prism AD'; also the right prism GJ' whose base GJ is a right section of the prism AD', and whose altitude is equal to a lateral edge AA' of prism AD'.
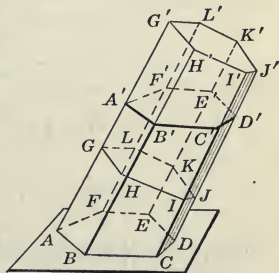
To prove prism $AD' =$ prism GJ'.

Outline of proof. Show that $AG = A'G'$, $BH = B'H'$, etc.; that the angles of face AH equal the angles of face $A'H'$, and that face $AH \cong$ face $A'H'$.

Likewise show face $BI \cong$ face $B'I'$, and face $AD \cong$ face $A'D'$.

Then   prism $AJ \cong$ prism $A'J'$.   § 656

Prism $AJ +$ prism $GD' =$ prism $A'J' +$ prism GD'.   Why?

∴ prism $AD' =$ prism GJ'.   Why?

EXERCISES

1. Compare the theorem of § 660 with the theorem of § 354.

2. If a wooden beam has a rectangular right cross section, show that if it is sawed lengthwise along a diagonal plane the two prisms formed are congruent.

3. In a truncated prism having a parallelogram for base, the sum of two opposite lateral edges is equal to the sum of the other two opposite lateral edges.

Prove $a + c = 2e$, and $b + d = 2e$.
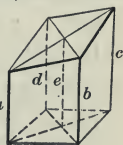
PARALLELEPIPEDS

661. A **parallelepiped** is a prism whose bases are parallelograms.

662. A **right parallelepiped** is a parallelepiped whose lateral edges are perpendicular to its bases.

663. An **oblique parallelepiped** is a parallelepiped whose lateral edges are oblique to its bases.

664. A **rectangular parallelepiped** is a right parallelepiped whose bases are rectangles.

665. A **cube** is a rectangular parallelepiped whose edges are all equal.
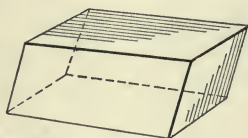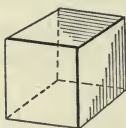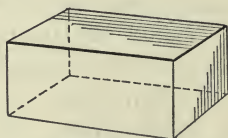
OBLIQUE   CUBE   RECTANGULAR
PARALLELEPIPED   PARALLELEPIPED

666. The **dimensions** of a rectangular parallelepiped are the lengths of the three edges drawn from one vertex. These dimensions are often called *length, breadth,* and *height.*

667. A **diagonal** of a parallelepiped is the line from any vertex to a vertex not in the same face.

668. The **volume** of any solid is the numerical measure of its magnitude in terms of some unit of measure.

The unit of measure usually taken is a cube one linear unit on an edge.

669. Ratio between Solids. By the ratio of one solid to another is meant the ratio of their numerical measures.

670. Prove the following facts about parallelepipeds:

(1) *All the faces (including the bases) of a parallelepiped are parallelograms.*

(2) *A parallelepiped is bounded by three pairs of parallel congruent parallelograms.*

(3) *A parallelepiped has three sets of four equal edges.*

(4) *Any two opposite faces of a parallelepiped may be taken as bases.*

(5) *All the faces of a rectangular parallelepiped are rectangles.*

(6) *All the faces of a cube are squares.*

671. Volume of Rectangular Parallelepiped. By a consideration analogous to that of § 344 the reasonableness of the following statement would be evident; here it is accepted without proof.

The volume of a rectangular parallelepiped is equal to the product of its three dimensions.

If V, a, b, and h are the numerical measures of the volume, length, breadth, and height, respectively, of any rectangular parallelepiped, then the above is stated in the formula

$$V = abh.$$

This means that, if the three dimensions are each measured in terms of some unit of length, then the volume is measured in terms of the corresponding unit of volume, which is a cube having an edge one linear unit in length.
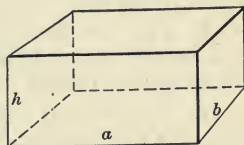
672. Theorem. *The volume of a rectangular parallelepiped is equal to the product of its base and altitude.*

673. Theorem. *The volume of a cube is equal to the cube of its edge.*

674. Theorem. *The volumes of two rectangular parallelepipeds are to each other as the products of their three dimensions.* $V : V' = abh : a'b'h'$.

675. Theorem. *The volumes of two rectangular parallelepipeds having equivalent bases are to each other as their altitudes.*

Since $\dfrac{V}{V'} = \dfrac{abh}{a'b'h'}$ and $ab = a'b'$, then $\dfrac{V}{V'} = \dfrac{h}{h'}$.

676. Theorem. *The volume of two rectangular parallelepipeds having equal altitudes are to each other as their bases.*

677. Theorem. *Two rectangular parallelepipeds having equivalent bases and equal altitudes are equal in volume.*

EXERCISES

1. Find the volumes of the following rectangular parallelepipeds: (1) 13 ft. by 27 ft. by 45 ft.   (2) 2 ft. 3 in. by 1 ft. 7 in. by 3 ft. 1 in.   (3) 30 ft. 6 in. by 41 ft. 6 in. by 12 ft.

2. A common brick is 8 in. by 4 in. by 2 in. Find the number of common brick in a pile 3 ft. by 6 ft. by 12 ft.

3. How many shoe boxes each 3 in. by 4 in. by 9 in. can be put in a packing box 3 ft. by 3 ft. 4 in. by 3 ft. 9 in.?

4. Find the number of cubic yards of earth to be excavated in digging a cellar 40 ft. by 26 ft. by 7 ft.

5. If 180 sq. ft. of zinc are required to line the bottom and sides of a cubical vessel, how many cubic feet of water will it hold?

6. A box car that is $36\frac{2}{3}$ ft. long and 8 ft. wide, inside measurements, can be filled with wheat to a height of $4\frac{1}{2}$ ft. Find how many bushels of wheat it will hold if $\frac{5}{4}$ cu. ft. are a bushel.

7. Find the cost at 40 cents a pound for sheet copper to line the bottom and sides of a cubical vessel 7 ft. on an edge, if the sheet copper weighs 12 oz. per square foot. Find volume.   *Ans.* $73.50; 81.45 bbl.

8. Find the cost of laying a stone wall 45 ft. long, 6 ft. high, and 2 ft. thick at $2.75 a perch. Use 22 cu. ft. for one perch.

9. The edge of a cube is 10 in. Find the edge of a cube which shall have a volume twice as great. Eight times as great.

10. Show that the edge and diagonal of a cube can be used as the two sides of a right triangle whose acute angles are 30° and 60°.

11. Are the diagonals of a rectangular parallelepiped perpendicular to each other? Are those of a cube? Prove.

12. The diagonals of a rectangular parallelepiped are equal.

13. The diagonals of a rectangular parallelepiped bisect each other.

14. Find the sum of all the face angles of a parallelepiped.

15. Find the edge of a cube if its volume is increased 200 cu. in. when each edge is increased 2 in.

16. The edges of a rectangular parallelepiped are a, b, and c. Find the length of a diagonal, and the entire area of the parallelepiped.

17. Test the following rule if 1 cu. in. of iron weighs 0.28 pounds: The weight of iron bars in pounds per foot of length equals the width in inches times the thickness in inches times $\frac{10}{3}$.

18. The sum of the squares of the four diagonals of any parallelepiped is equal to the sum of the squares of the twelve edges. Apply Ex. 11, p. 181.

678. Theorem. *The volume of any parallelepiped is equal to the product of its base and altitude.* $V = Bh$.
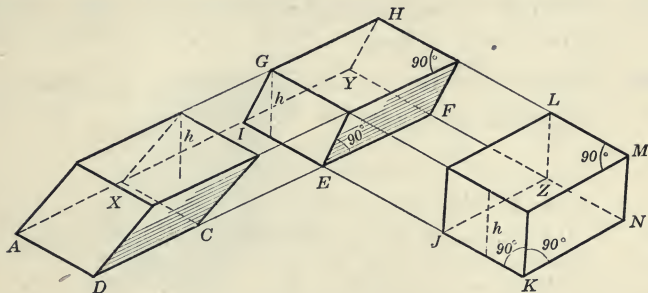
Given X an oblique parallelepiped, with area of base $AC = B$, and height h.

To prove $V = Bh$, where V denotes volume.

Proof. Extend DC and the other edges of X that are parallel to it.

On DC extended, take $EF = DC$.

Pass planes EG and FH through E and F and perpendicular to DC, forming a right parallelepiped Y, in which EG is a right section.

Then parallelepiped $X =$ parallelepiped Y.   § 660

Extend IE and the other edges of Y that are parallel to it.

On IE extended, take $JK = IE$.

Pass planes JL and KM through J and K and perpendicular to IE, forming a rectangular parallelepiped Z.

Then parallelepiped $Y =$ parallelepiped Z.   § 660

Therefore parallelepiped $X =$ parallelepiped Z.   § 104

But volume of $Z =$ base $JN \times NM$.   § 672

In which $NM = h$.   Why?

And $\square JN = \square IF = \square AC = B$.   Why?

                  $\therefore V = Bh$.   Why?

679. Theorem. *Parallelepipeds having equivalent bases and equal altitudes are equivalent.*

680. Theorem. *Two parallelepipeds are to each other as the products of their bases and altitudes.* ·

681. Theorem. *Two parallelepipeds having equal altitudes are to each other as their bases.*

682. Theorem. *Two parallelepipeds having equivalent bases are to each other as their altitudes.*

EXERCISES

1. Compare theorems of §§ 678-682 with those of §§ 355, 358-361.

2. Use the notation of § 678, and show that in any parallelepiped $B = \dfrac{V}{h}$, and $h = \dfrac{V}{B}$.

3. Find the length of the diagonal of a rectangular parallelepiped whose dimensions are 30 ft., 40 ft., and 12 ft.

4. A parallelepiped has a base in the form of a rhombus whose edges and one diagonal are each 10 in. Find the volume if the altitude is 8 in.

5. A parallelepiped has a rectangular base 8 in. by 15 in., and square ends. Find its volume if one of the sides is a parallelogram having an angle of 60°.

6. Find the diagonal of a cube whose volume is 512 cu. in.

7. Find the dimensions of a rectangular parallelepiped having a volume of 12,960 cu. in., if the dimensions are in the ratio of 3 : 4 : 5.

                                            Ans.   18 in., 24 in., 30 in.

8. Find the edge of a cube whose surface and volume have the same numerical value.

9. A parallelepiped of altitude 8 in. has the same volume as a cube that is 12 in. on an edge. Find the area of the base and the dimensions of the base if it is a parallelogram having an angle of 60° and one side equal to 18 in.

10. Find the length of the bar that can be made from 1 cu. ft. of steel, if the bar has a rectangular cross section $\frac{1}{2}$ in. by $1\frac{1}{2}$ in.

11. Find the volume to the nearest .001 cu. ft. of a rectangular tank 17 ft. 2 in. by 19 ft. 3 in. by 3 ft. 7 in.   *Ans.*   1184.142 cu. ft.

12. Show geometrically that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

13. The total area of a right prism whose base is a rectangle is 118 sq. ft., the volume is 70 cu. ft., and the altitude is 7 ft. Find dimensions of the base.

VOLUMES OF PRISMS AND CYLINDERS

683. Theorem. *The plane passed through the two diagonally opposite edges of a parallelepiped divides the parallelepiped into two equivalent triangular prisms.*
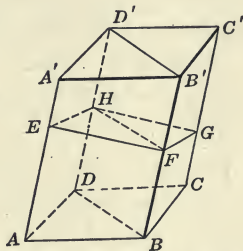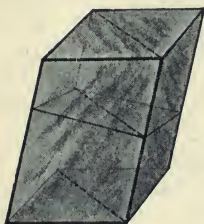
Given the parallelepiped AC', divided into two triangular prisms A'-ABD and C'-BCD by the diagonal plane passing through the edges BB' and DD'.

To prove prism A'-ABD = prism C'-BCD.

Proof. Let $EFGH$ be a right section of the parallelepiped, cutting the diagonal plane in FH.

Face $AB' \parallel$ face DC', and face $AD' \parallel$ face BC'. Why?

Then $EF \parallel HG$, and $EH \parallel FG$. § 562

And $EFGH$ is a parallelogram. Why?

FH is a diagonal of $\square EFGH$. Why?

Hence $\triangle EFH \cong \triangle FGH$. § 154

Prism A'-ABD equals a right prism with base EFH and altitude AA'. § 660

Prism C'-BCD equals a right prism with base FGH and altitude AA'. Why?

But these two right prisms are congruent and so equivalent.

    ∴ prism A'-ABC = C'-BCD. § 104

EXERCISES

1. Find the edge of a cube that is increased in volume 127 cu. in. when its edge is increased 1 in.

2. A rectangular solid with a square base has a volume of 80 cu. ft., and a surface of 112 sq. ft. Find its dimensions. *Ans.* 4 ft. by 4 ft. by 5 ft.

684. Theorem. *The volume of a triangular prism is equal to the product of its base and altitude.* $V = Bh$.

Given the triangular prism A'-ACD.

To prove $V = Bh$, where V denotes volume, B area of base, and h altitude.

Suggestion. Complete the parallelepiped A'-$ACED$. Show that prism A'-$ACD = \frac{1}{2}$ parallelepiped A'-$ACED$. Apply § 683.
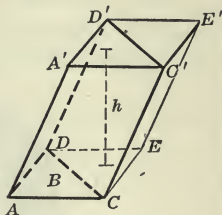
685. Theorem. *The volume of any prism is equal to the product of its base and altitude.* $V = Bh$.

Given any prism AD'.

To prove $V = Bh$, where V denotes volume, B area of base, and h altitude.

Suggestion. Show that the prism can be divided into triangular prisms by diagonal planes. Add the prisms thus formed.
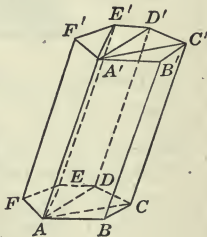
686. Theorem. *Prisms having equivalent bases and equal altitudes are equivalent.*

687. Theorem. *Prisms having equivalent bases are to each other as their altitudes.*

688. Theorem. *Prisms having equal altitudes are to each other as their bases.*

EXERCISES

1. Find the volume of a prism whose base is a regular hexagon 6 in. on a side, and whose altitude is 24 in.

2. In a prism given $V = 226$ and $B = 43.6$; find h.

3. The sides of a right section of a triangular prism are 4 in., 5 in., and 7 in. Find the volume if a lateral edge is 16 in.

4. The cost of digging a ditch, including all expenses and profits, is estimated at 27 cents a cubic yard. Find the cost of digging a ditch 15 mi. long, 10 ft. wide at the bottom, 20 ft. at the top, and 6 ft. deep. The cross section is a trapezoid. *Ans.* $71,280.

5. Show that an oblique prism of wood may be changed into a right prism by cutting along a right section and interchanging the two parts.

6. One of the concrete pillars to support a floor in a concrete building is 12 ft. high and has as a cross section a regular hexagon 8 in. on a side. Find its weight if the concrete weighs 138 lb. per cubic foot.   *Ans.* 1912 lb.

7. A flow of 300 gallons per second will supply water for a stream of what depth, if the stream is 4 ft. wide and flows 5 miles per hour?

Ans. $16\frac{13}{32}$ in.

8. Find the number of cubic yards of crushed rock to make a road 1 mile in length and of cross section as shown in the figure.
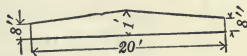
9. One cubic inch of steel weighs 0.29 lb. An I-beam has a cross section as shown in the figure and a length of 22 ft. Find its weight.
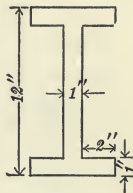
10. An iron casting shrinks $\frac{1}{8}$ in. per linear foot in cooling down to 70 degrees Fahrenheit. How many cubic inches is the shrinkage per cubic foot?

Ans. 53.44 cu. in.

11. How many cubic yards of soil will it take to fill in a lot 50 ft. by 100 ft. if it is to be raised 3 ft. in the rear and gradually sloped to the front where it is to be $1\frac{1}{2}$ ft. deep?   *Ans.* $416\frac{2}{3}$ cu. yd.

12. In a regular triangular prism, the edge of the base and the altitude are equal. Find these dimensions if the volume is $128\sqrt{3}$ cu. in.

13. In a regular triangular prism, the altitude is 6 in. more than the edge of the base. Find the dimensions if the volume is $224\sqrt{3}$ cu. in.

14. Find the volume of a regular hexagonal prism whose base has an area of 37.5 sq. ft., and whose altitude equals an edge of the base.
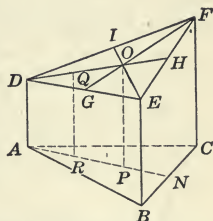
15. The perpendicular drawn to the lower base of a truncated triangular prism from the intersection of the medians of the upper base, equals one third the sum of the three lateral edges.

16. The volume of any oblique prism is equal to the product of the area of a right section by the length of a lateral edge.

17. The volume of a regular hexagonal prism is $30\sqrt{3}$ cu. in. and its lateral area is 180 sq. in. Find its altitude and base edge.

18. The volume of a triangular prism is equal to the area of a lateral face times one-half the perpendicular drawn to that face from the opposite edge.

SOLID GEOMETRY

689. Theorem. *The volume of a circular cylinder is equal to the product of its base and altitude.* $V = Bh.$

Given a circular cylinder.

To prove that $V = Bh$, where V denotes volume, B area of base, and h altitude.

Proof. Inscribe a prism, whose base is a regular polygon, in the cylinder, and let V' denote its volume and B' the area of its base.
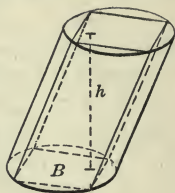
By doubling indefinitely the number of faces of the prism,

$$V' \to V, \text{ and } B' \to B. \qquad \text{§§ 651 (3), 489}$$

Then $\qquad\qquad B'h \to Bh. \qquad\qquad$ § 485 (2)

But $\quad V' = B'h$, being variables that are always equal. § 685

$$\therefore V = Bh. \qquad\qquad \text{§ 485 (1)}$$

EXERCISES

1. Show that the volume of a hollow cylinder with outer diameter D, inner diameter d, and altitude h, is given by the formula,

$$V = \tfrac{1}{4}\pi h(D^2 - d^2) = \tfrac{1}{4}\pi h(D+d)(D-d). \quad \text{(See Ex. 8, p. 249.)}$$

2. A cylindrical oil tank 3 ft. in diameter and 10 ft. long will contain how many gallons? (1 gal. = 231 cu. in.) *Ans.* 528.8.

3. The outer diameter of a hollow cast iron shaft is 18 in.; and its inner diameter is 10 in. Calculate its weight if the length is 20 ft. and cast iron weighs 0.26 lb. per cubic inch.

4. A peck measure is to have a diameter of 8 in. How deep should it be? (1 bu. = 2150.42 cu. in.)

5. Water is flowing at the rate of 10 miles per hour through a pipe of diameter 16 in. into a rectangular reservoir 197 yd. long and 87 yd. wide. Calculate the time in which the surface of the water in the tank will be raised 3 in. *Ans.* 31.38 minutes.

6. A certain handbook gives the following "rules of thumb" for finding the volume in gallons of a cylindrical tank:

(1) $V = (\text{diameter in feet})^2 \times 5\tfrac{7}{8} \times (\text{height in feet}).$

(2) $V = (\text{diameter in feet})^2 \times \tfrac{1}{2} (\text{height in inches})$ less 2% of the product. Find the per cent of error for each rule.

7. Find the height of a 10-gallon wash boiler whose base is 10 in. wide with semicircular ends, the length of the straight part of the sides being $9\tfrac{1}{4}$ in. *Ans.* 13.5 in.

8. A cylinder to cool lard is 4 ft. in diameter and 9 ft. long and makes four revolutions per minute. At each revolution, the hot lard cools upon the surface to a depth of $\frac{1}{8}$ in. How many pounds of lard will it cool in one hour if 1 cu. ft. of lard weighs $56\frac{1}{4}$ pounds?

9. If a tank 5 ft. in diameter and 10 ft. deep holds 10,000 pounds of lard, what will be the depth of a tank of 2000 pounds capacity if its diameter is 3 ft.? If this tank has a jacket around it on the bottom and sides 3 in. from the surface of the tank, how many gallons of water will the space between the jacket and the tank hold?

10. A conduit made of concrete has a cross section with dimensions as shown in the figure. How many cubic yards of concrete are used in making one mile of this conduit?
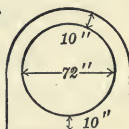
11. The cylindrical water tower shown in the figure is at Long Beach, N.Y. Its diameter is 34 ft., height 150 ft., and it is said to have a capacity of 1,020,000 gallons. Is the capacity given correct?

12. Does a cylindrical water tank 42 in. in diameter and 14 ft. long hold 1000 gallons?

13. The following "rule of thumb" is used for finding the weight of round iron. The weight of round iron in pounds per foot equals the square of the diameter in quarter inches divided by 6. Find the per cent of error in using the rule if iron weighs 0.28 lb. per cubic inch.

14. How much water will a horizontal steam boiler 5 ft. in diameter and 16 ft. long with 70 tubes of diameter 3 in. running lengthwise, hold if one third of the volume is for steam?

15. Find the height of a cylindrical oil tank with a diameter of 16 in. to hold one barrel.   *Ans.*   36.2 in.
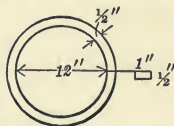
16. A tool steel ring for a steam cylinder is forged from round stock 3 in. in diameter. Find the length of stock to make a ring with the dimensions given in the figure.

17. The segment in the figure is a counter-balance $5\frac{1}{2}$ in. thick. Find its weight if made of cast-iron weighing 0.26 lb. per cubic inch.   *Ans.*   228.5 lb.

18. Find the weight of a hollow hexagonal bar 16 ft. long and weighing 0.28 lb. per cubic inch. The cross section is a regular hexagon $1\frac{1}{4}$ in. on a side, with a circle $1\frac{1}{4}$ in. in diameter at the center.   *Ans.* $152\frac{1}{4}$ lb.
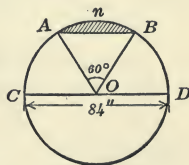
SIMILAR PRISMS AND CYLINDERS

690. Definitions. Two right prisms having similar polygons for bases, and whose altitudes are in the same ratio as two corresponding base edges, are **similar**.

Two right circular cylinders are **similar** if their altitudes are in the same ratio as the radii of their bases.

691. Theorem. *The volumes of two similar right prisms are in the same ratio as the cubes of corresponding base edges.*
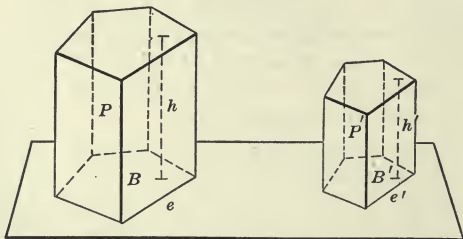
Given two similar right prisms P and P', having altitudes h and h', bases B and B', and two corresponding base edges e and e'

To prove $\dfrac{V}{V'} = \dfrac{h^3}{h'^3} = \dfrac{e^3}{e'^3}$.

Proof. $\qquad\qquad V = Bh,$ and $V' = B'h'.$ $\qquad\qquad$ § 685

Hence $\qquad\qquad \dfrac{V}{V'} = \dfrac{Bh}{B'h'} = \dfrac{B}{B'} \cdot \dfrac{h}{h'}.$ $\qquad\qquad$ § 108

But $\qquad\qquad\qquad \dfrac{B}{B'} = \dfrac{e^2}{e'^2}.$ $\qquad\qquad$ § 446

And $\qquad\qquad\qquad \dfrac{h}{h'} = \dfrac{e}{e'}.$ $\qquad\qquad$ § 690

$\qquad\qquad \therefore \dfrac{V}{V'} = \dfrac{e^2}{e'^2} \cdot \dfrac{e}{e'} = \dfrac{e^3}{e'^3}.$ $\qquad\qquad$ § 111

Also $\qquad\qquad\qquad \dfrac{V}{V'} = \dfrac{h^3}{h'^3}.$ $\qquad\qquad$ § 111

692. Theorem. *The volumes of two similar right circular cylinders are in the same ratio as the cubes of their altitudes, radii, or diameters.*
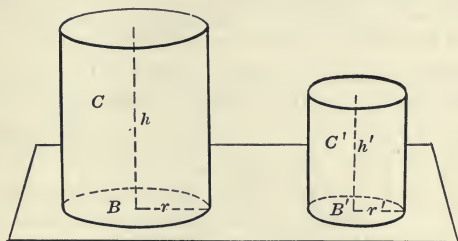
Given two similar right circular cylinders C and C', having volumes V and V', bases B and B', altitudes h and h', radii r and r', and diameters d and d'.

To prove $\dfrac{V}{V'} = \dfrac{h^3}{h'^3} = \dfrac{r^3}{r'^3} = \dfrac{d^3}{d'^3}$.

The proof is similar to the preceding.

EXERCISES

1. Using the notation of § 691 for similar prisms, also S and S' for lateral areas, and T and T' for total areas, prove the following:

$$(1)\ \frac{S}{S'} = \frac{e^2}{e'^2} = \frac{h^2}{h'^2}. \qquad (2)\ \frac{T}{T'} = \frac{e^2}{e'^2} = \frac{h^2}{h'^2}.$$

2. Using the notation of § 692 for similar cylinders, also S, S', T, and T' as in Exercise 1, prove the following:

$$(1)\ \frac{S}{S'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}. \qquad (2)\ \frac{T}{T'} = \frac{h^2}{h'^2} = \frac{r^2}{r'^2} = \frac{d^2}{d'^2}.$$

3. Find the ratios of the volumes, the lateral areas, and the total areas of two similar right prisms having hexagonal bases two of whose corresponding edges are 2 in. and 5 in. respectively.

4. Two similar right circular cylinders have radii of 3 in. and 7 in. respectively. Find the ratio of their volumes. Of their lateral areas. Of their total areas.

5. Two similar right circular cylinders have bases whose areas are 20.25 sq. in. and 100 sq. in. respectively. Find the ratio of their volumes.

6. The total areas of two similar right circular cylinders are 625 sq. in. and 324 sq. in. respectively. Find the diameter of the second if the diameter of the first is $6\frac{1}{4}$ in.

7. A rectangle having dimensions of 10 in. and 15 in. is revolved, first about the side that is 10 in., and second, about the side that is 15 in. Find the ratio of the volumes of the two cylinders formed.

8. The number of feet of lumber in a log is often based upon a standard log usually 12 ft. long and 24 in. in diameter inside the bark at the small end. If v, d, and l are the volume, diameter, and length respectively of the standard log; and V, D, and L, the corresponding measurements of the log to be measured, then $V = \dfrac{vD^2L}{d^2l}$. Derive this formula.

QUESTIONS

1. What is a prismatic surface? A cylindrical surface? Is the directrix necessarily closed?

2. In § 617, suppose the directrix is coplanar with the generatrix, what is the form of the surface generated?

3. What are the formulas or rules for finding areas of prisms and cylinders? What are the formulas or rules for finding the volumes of each of these solids?

4. Why is there such a close relation between theorems concerning prisms and cylinders?

5. Can you find the area of an oblique circular cylinder? Of an oblique prism? Can you find the volume of each of these?

6. Trace the steps in finding the volume of a parallelepiped.

7. Trace the steps in finding the volume of any prism. Of a cylinder.

8. What effect does it have upon the volume of a prism or cylinder if the base is doubled? If the altitude is doubled? If both base and altitude are doubled?

9. What effect does it have upon the lateral area of a right circular cylinder if the circumference of the base is doubled? If the area of the base is doubled?

10. Which do you consider the more common form in nature, the cylinder or the prism? Name some examples of each form.

11. Which is the more common of these forms in buildings and architecture? In machinery? Give illustrations of triangular prisms, of quadrangular prisms, of pentagonal, of hexagonal, etc.

GENERAL EXERCISES

COMPUTATIONS

1. Find the number of barrels each of the following cylindrical tanks will hold: (1) diameter 5 ft. and depth 5 ft., (2) diameter 20 ft. and depth 19 ft. (1 bbl. = 3.211 cu. ft.)

2. Find the cost of laying the stonework, at $1.75 per cubic yard, in two abutments for a bridge, each abutment to be 8 ft. high, $3\frac{1}{2}$ ft. thick, 20 ft. long at the bottom, and 15 ft. at the top.

3. Find the cost of common brick in the pier with a cross section as shown in the figure, and a height of 12 ft. 6 in., at $7.00 per thousand. Count 20 brick to 1 cu. ft.   *Ans.* $24.15.
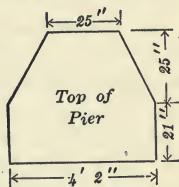
4. A stream flowing 5 miles per hour must be how large in cross section to supply water to a depth of $1\frac{1}{2}$ in. per week, for 160 acres of land?   *Ans.* $28\frac{2}{7}$ sq. in.

5. A water tank in a Pullman car has a vertical section as shown in the figure, and a length of 52 in. Find its capacity in gallons.   *Ans.* 68.3 gal.

Consider the arc as a part of a circle and apply formula (1) of § 509.
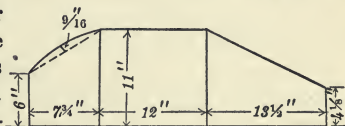
6. The flanges at the joining of two ends of flanged steam pipes 10 in. in inside diameter are bolted together by 14 bolts $\frac{3}{4}$ in. in diameter. If the pressure in the pipes is 200 pounds per square inch, find what each bolt must hold. How much is this per square inch cross section of the bolts? Suppose that the bolts have 10 pitch U.S. standard thread. This makes the root diameter 0.620 in.
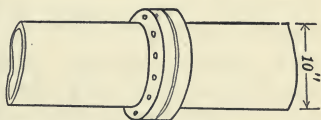
Ans. 1122 lb.; 3718 — lb.

7. Find the diameter of a cylindrical oil tank 40.5 in. high that is to hold 1 barrel.   *Ans.* $15\frac{1}{8}$ in.

8. Representing the dimensions of a rectangular solid by x, y, and z, find their values if when each is increased 2 in. the volume is increased 150 cu. in., the face of dimensions x and y is increased 18 sq. in., and the total surface is increased 110 sq. in.   *Ans.* 3 in., 4 in., and 5 in.

9. Two cubes whose edges differ by 1 in. have volumes that differ by 397 cu. in. Find the edges of each cube.

10. A rectangular sheet of tin, 12 in. by 16 in., is made into an open box by cutting out a square from each corner and turning up the sides. Find the size of the square cut out if the volume of the box is 180 cu. in.
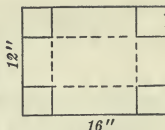
11. A rectangular piece of tin a inches longer than it is wide is made into an open box, containing c cubic inches, by cutting from each corner a square of side b inches. Find the dimensions of the original piece of tin.

12. One edge of a rectangular box is increased 6 in., another 3 in., and the third 4 in., making a cube whose volume is 862 cu. in. greater than that of the original box. Find the dimensions of the box.

Ans. 4.14 in. by 7.14 in. by 6.14 in.

13. If in a right prism the altitude is equal to a side of the base, find the volume if the base is an equilateral triangle whose sides are a.

14. Find the capacity in gallons of a water tank for a locomotive tender. The dimensions are as given in the figure for the length cross section, and the width is 9 ft. 6 in.
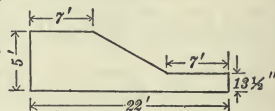
15. A certain coal and coke company finds it necessary to construct a wharf wall 150 ft. long, and having a cross section with dimensions as shown in the figure. Find the number of cubic yards in the wall.

By "batter 1:12" is meant that the slant is 1 ft. in a vertical rise of 12 ft.

16. In a right prism whose volume is 54, the lateral area is four times the area of the base which is an equilateral triangle. Find the edge of the base. *Ans.* 6.

17. The total areas of two similar cylinders of revolution are 75 sq. in. and 192 sq. in. respectively. If the volume of the first cylinder is 250 cu. in., what is the volume of the second?

18. In a right circular cylinder, given $V = \pi r^2 h$ and $T = 2\pi r^2 + 2\pi rh$. Find T in terms of V and h.

Ans. $T = \dfrac{2V}{h} + 2\sqrt{V\pi h}$.

19. In a right circular cylinder, find V in terms of the circumference c of the base and the total area T.

Ans. $V = \dfrac{2\pi cT - c^3}{8\pi^2}$.

20. Find the steam capacity of a horizontal cylindrical boiler 4 ft. in diameter and 16 ft. long, if the height of the segment occupied by the steam is 18 in. (Use (2) of §509).
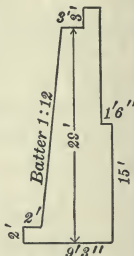
21. A cylindrical tank of diameter 30 in. and 34 in. long rests on its side. Find the number of gallons of gasoline in the tank if the depth is $5\frac{1}{2}$ in. *Ans.* 13 gal. nearly.

22. The volume of an irregular shaped body is often found by immersing it in water and determining the amount of water displaced. A cylindrical vessel that has a diameter of 4 in. is partly filled with water. A stone immersed in the water raises its level $3\frac{1}{2}$ in. Find the volume of the stone.

23. A silo is used to keep fodder in a green and succulent state for feeding farm animals. It is usually built in a cylindrical form.

Find the capacity in tons of a silo in the form of a right circular cylinder 20 ft. in diameter and 32 ft. high, if a cubic foot of silage weighs 40.7 lb.

24. An 18 acre field yields 11.5 tons of silage per acre. What must be the height of a cylindrical silo 20 ft. in diameter to hold all the silage if 1 cu. ft. of silage weighs 38.4 lb.?

25. A silo is in the form of a right circular cylinder, and is 20 ft. in diameter inside and 32 ft. high. How many cubic yards of concrete did it take to build it, if the floor and wall are each 6 in. thick, and the foundation wall is 8 in. thick and 5 ft. deep?

26. The connecting rod with dimensions as given in the figure is made from stock with the dimensions shown. Find length of stock that it is necessary to allow for the cylindrical part of the rod.

Connecting Rod

Stock
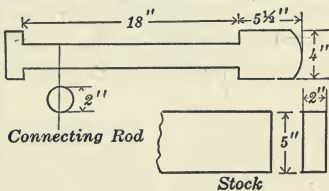
27. Find the volume of the beveled washer with dimensions as shown in the figure.

28. A pine log 2 ft. in diameter and 16 ft. long is floating in water. Find the weight of the log if two-thirds of the volume of the log is under water. (Water weighs 62.5 lb. per cubic foot.)
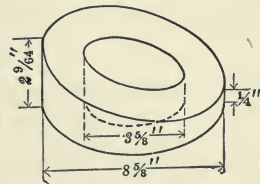
29. Will a floating pine log $1\frac{1}{2}$ ft. in diameter and 10 ft. long support a man weighing 180 pounds if the specific gravity of the log is 0.72?

30. The following is the record of a test in which a high speed drill removed 70.56 cu. in. of cast-iron per minute. The penetration per minute was $57\frac{1}{2}$ in., the feed $\frac{1}{10}$ in. per turn, and the number of turns per minute was 575. Do these numbers agree?

THEOREMS AND PROBLEMS

1. Derive the following formulas for finding the volume, V, of a hollow circular cylinder of length h and cross sectional dimensions as given in the figure:

(1) $V = \pi h (R+r)(R-r)$.

(2) $V = \frac{1}{2} \pi h t (D+d)$.

(3) $V = \pi h t (d+t)$.

(4) $V = \pi h t (D-t)$.

2. Construct a plane through a point and tangent to a given right circular cylinder.

3. If the radius of one cylinder is equal to the altitude of a second, and the radius of the second is equal to the altitude of the first, what is the ratio of their volumes?

4. The intersection of two planes each tangent to a circular cylinder is parallel to the elements of the cylinder.

5. Prove that the volume of a right circular cylinder is equal to its lateral area times one-half the radius of its base.

6. The volume of two right circular cylinders are equal. Write a proportion between their lateral areas and their radii.

7. If a straight line has more than two points common to the curved surface of a right circular cylinder, the line is an element of the surface.
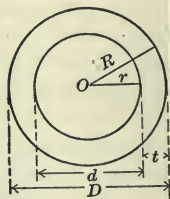
CHAPTER VIII

PYRAMIDS AND CONES

693. A moving straight line that always contains a fixed point, and always intersects a given straight line, generates a **plane.** Why? § 542

694. Pyramidal Surface. A moving straight line that always contains a fixed point, and always intersects a broken line not coplanar with it, generates a **pyramidal surface.**
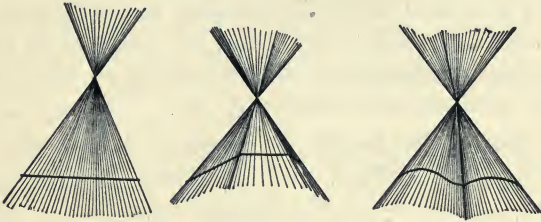
695. Conical Surface. A moving straight line that always contains a fixed point, and always intersects a plane curved line not coplanar with it, generates a **conical surface.**

696. The fixed point is called the **vertex** of the pyramidal, or conical, surface.

697. The two parts of the pyramidal, or conical, surface on opposite sides of the vertex are called **nappes.**

The words *generatrix, directrix, element,* and *closed surface* have the same significance here as in §§ 618, 619.

The directrix is not necessarily closed, but in this text only those that are closed are considered.

The word "section" has the same significance as before (§ 620), however, the cutting plane must not pass through the vertex.

343

698. A **pyramid** is the solid formed by cutting all the elements of one nappe of a closed pyramidal surface by a plane.

699. A **cone** is the solid form by cutting all the elements of one nappe of a closed conical surface by a plane.
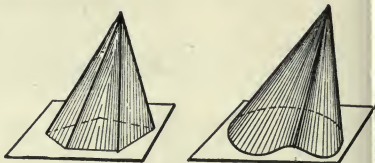
The meanings of the words: *base, lateral face, lateral edge, base edge, lateral area, total area,* are apparent from the definitions of §§ 623, 624.

700. The **altitude** of a pyramid, or cone, is the perpendicular distance from the vertex to the base.

701. Pyramids Classified According to the Number of Lateral Faces. Pyramids, like prisms, are classified as **triangular, quadrangular, pentagonal,** etc., according as their bases are *triangles, quadrilaterals, pentagons,* etc.

702. Regular Pyramids. A pyramid whose base is a regular polygon, and whose altitude meets the center of its base, is called a **regular pyramid.**

The altitude of a regular pyramid is called its **axis.**

The altitude of one of the lateral faces of a regular pyramid is called the **slant height.**

In the figure, OV is the altitude, BV a lateral edge, and NV the slant height.

REGULAR PYRAMID
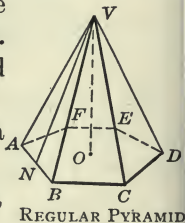
703. Circular Cones. A cone whose base is a circle is called a **circular cone.**

The line joining the vertex of a circular cone to the center of its base is called the **axis** of the cone.

704. A **right circular cone** is a circular cone whose axis is perpendicular to its base.

CIRCULAR CONE

RIGHT-CIRCULAR CONE
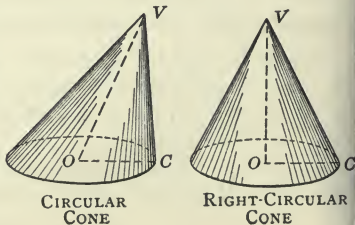
The radius of a circular cone is the **radius** of its base.

Since a right circular cone may be generated by revolving a right triangle about one of its sides as an axis it is often called a **cone of revolution.**

The length of an element of a right circular cone is its **slant height.**
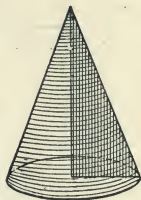
705. Prove the following facts concerning cones and pyramids:

(1) *The lateral faces of a regular pyramid are congruent isosceles triangles.*

(2) *The lateral edges of a regular pyramid are equal.*

(3) *The elements of a right circular cone are equal.*

(4) *The axis of a right circular cone coincides with its altitude.*

(5) *A straight line drawn from the vertex of a cone to any point in the perimeter of its base is an element.*

(6) *The section of a circular cone made by a plane containing an element is a triangle.*

(7) *The section of a pyramid made by a plane through its vertex is a triangle.*

EXERCISES

1. Find an element of a right circular cone whose altitude is 15 and radius 7.

2. Find the altitude of a right circular cone whose slant height is s and radius r.

3. Find the altitude of a regular pyramid each face and the base being an equilateral triangle 10 in. on a side.

4. Into how many parts do the nappes of a conical surface divide space?

5. Why is it stated in the definition of a pyramidal surface that the vertex and directrix must not be coplanar?

6. The altitude of a right circular cone is 10 in. and the radius of the base is 6 in. Find the area of a section made by a plane passing through the vertex and 3 in. from the center of the base. In the figure find the area of VAB.
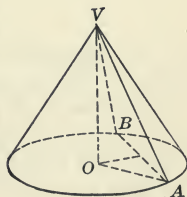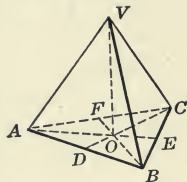
706. Theorem. *The lateral edges and the altitude of a pyramid, or the elements and the altitude of a cone, are divided proportionally by a plane parallel to its base.*
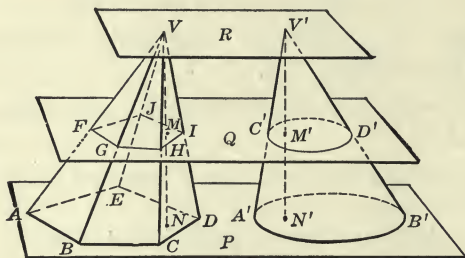
Pass a plane through the vertex parallel to the base and apply § 568.

707. Theorem. *The section of a pyramid made by a plane parallel to the base is similar to the base.*

Given pyramid *V-AD*, cut by plane *P* parallel to base *AD*, forming the section *FI*.

To prove polygon *FI* ⌢ polygon *AD*.

Proof. $\triangle FVG \backsim \triangle AVB$, $\triangle GVH \backsim \triangle BVC$, etc.   Why?

Then $\dfrac{FG}{AB}=\dfrac{VG}{VB}$, and $\dfrac{GH}{BC}=\dfrac{VG}{VB}$, etc.   § 428

Hence $\dfrac{FG}{AB}=\dfrac{GH}{BC}=\dfrac{HI}{CD}=\dfrac{IJ}{DE}=\dfrac{JF}{EA}.$ Why?

Also $\angle JFG = \angle EAB$, $\angle FGH = \angle ABC$, etc.   § 567

            ∴ polygon *FI* ⌢ polygon *AD*.   §§ 441, 431
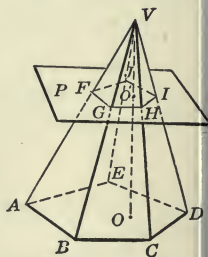
EXERCISES

1. The lateral edges of a pyramid are respectively 15 in., 12 in., and 16 in. Find the parts of each made by a plane that is parallel to the base and which divides the altitude into parts that are in the ratio of 1 : 3.

2. The area of the base of a pyramid is 125 sq. in. Find the area of a section 8 in. from the base and parallel to it, if two corresponding edges of the base and the section are respectively 10 in. and 8 in.

708. Theorem. *The section of a circular cone made by a plane parallel to its base is a circle, the center of which is the point where the axis intersects it.*

Given a circular cone V-ABC, with section EH made by the plane P parallel to the base. Also given axis VO piercing plane P in O'.

To prove that EH is a circle having O' as center.

Proof. Take F and G any two points in EH.

Then the planes determined by $FO'V$ and $GO'V$ intersect the plane of the base in OB and OC, the plane P in $O'F$ and $O'G$, and the lateral surface in the elements VB and VC, respectively.   Why?
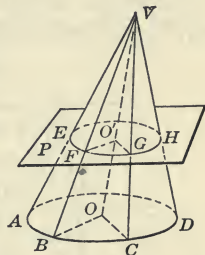
Prove that $\triangle FO'V \sim \triangle BOV$, and $\triangle GO'V \sim \triangle COV$.

Then $\dfrac{FO'}{BO} = \dfrac{GO'}{CO}.$ Why?

But $BO = CO.$ Why?

Therefore $FO' = GO'$ Why?

Since F and G are any two points in the section EH, all points in this section are equally distant from O'.

∴ section EH is a circle with center O'.   § 263

709. Theorem. *The area of any section of a pyramid, or circular cone, parallel to the base is to the area of the base as the square of its distance from the vertex is to the square of the altitude of the pyramid, or cone.*

EXERCISES

1. The area of the base of a pyramid is 64 sq. in. and its altitude is 20 in. Find the area of a section parallel to the base and 4 in. from the vertex.

2. The diameter of the base of a circular cone is 6 in. and the altitude is 10 in. Find the area of a section parallel to the base and 6 in. from the vertex.

710. Theorem. *If two pyramids, two circular cones, or a pyramid and a circular cone, have equal altitudes and bases of equal area, sections made by planes parallel to the bases and at equal distances from the vertices, have equal areas.*
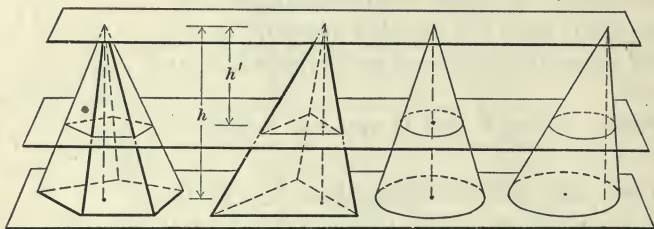
Suggestion. Let the equal altitudes be h and the distance from the vertices to the sections h'.

Then area of any section is to area of base as $h'^2 : h^2$.   § 709

But it is given that the bases are equal, and the altitudes equal.

Therefore the sections are equal in area.   Why?

EXERCISES

1. In the figure of § 707, $VO' = 8$, $O'O = 12$. $VB = 25$, and $VA = 27$; find VG and VF.

2. If the base of the pyramid of § 707 has an area of 180 sq. in., using the lengths given in Exercise 1, find the area of the section.

3. The area of the base of a pyramid is 150 sq. cm. and its altitude is 2 dm. Find the area of a section 8 cm. from the vertex.

4. The areas of parallel sections of a pyramid, or of a circular cone, are to each other as the squares of their distances from the vertex.

5. The altitude of a pyramid is h. At what distance from the vertex must a plane be passed parallel to the base so that the section shall be (1) one-half as large as the base? (2) One-third? (3) One-fifth?

6. Each side of the base of a regular hexagonal pyramid is 8 in. If the altitude of the pyramid is 20 in., how far from the vertex must a plane parallel to the base be so that the area of the section shall be $24\sqrt{3}$ sq. in.?

7. The area of the base of a cone is 216 sq. in. and its altitude is 24 in. Find the distance between two sections parallel to the base, which have areas of 144 sq. in. and 72 sq. in., respectively.

AREAS OF PYRAMIDS AND CONES

711. Theorem. *The lateral area of a regular pyramid is equal to half the product of the perimeter of the base and the slant height.* $S = \frac{1}{2}ps.$

Given the regular pyramid $V\text{-}ABCDE.$

To prove $S = \frac{1}{2}ps$, where S denotes lateral area, s slant height, and p perimeter of base.

Proof. The lateral faces are congruent triangles.   Why?

The area of each face triangle $= \frac{1}{2}s \times$ its base.
                                                        Why?

The sum of the bases of the triangles $= p$.   Why?

Hence the sum of the areas of the faces $= \frac{1}{2}ps$.   Why?

$$\therefore S = \frac{1}{2}ps.$$
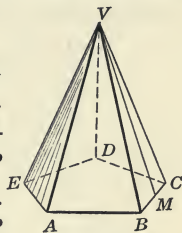
EXERCISES

1. Using the notation of §711 show that $s = \dfrac{2S}{p}$, and $p = \dfrac{2S}{s}$.

2. Find the lateral area and the total area of a regular hexagonal pyramid whose altitude is 8 in. and base edge is $4\sqrt{3}$ in.

3. Find the lateral area and the total area of a regular triangular pyramid having base edges equal to 8 in. and lateral edges equal to 12 in.

712. Plane Tangent to a Cone. A plane is **tangent to a cone** if it meets the conical surface in one element only.

713. Theorem. *If a plane is tangent to a circular cone, its intersection with the plane of the base is tangent to the base.*

Proof similar to that of § 642.

714. Theorem. *The plane determined by the tangent to the base of a circular cone and the element through the point of contact, is tangent to the cone.*

Compare with theorem of § 643.
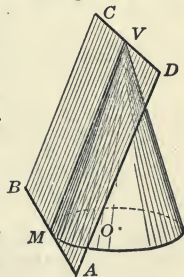
715. Inscribed and Circumscribed Pyramids. A pyramid is said to be **inscribed in a cone** if the pyramid and cone have the same vertex, and if the base of the pyramid is inscribed in the base of the cone. The cone is then said to be **circumscribed** about the pyramid.
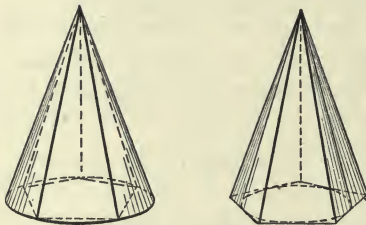
716. A pyramid is said to be **circumscribed about a cone** if the pyramid and cone have the same vertex, and if the base of the pyramid is circumscribed about the base of the cone. The cone is then said to be **inscribed** in the pyramid.

It is evident that the lateral edges of an inscribed pyramid are elements of the cone; and that the lateral faces of the circumscribed pyramid are tangent to the cone by §§ 712, 715.

717. Area of Cone, Practical Consideration. If the lateral surface of a right circular cone is covered with paper, this paper when peeled from the cone can be spread out and will be a sector of a circle, whose radius is the slant height of the cone, and
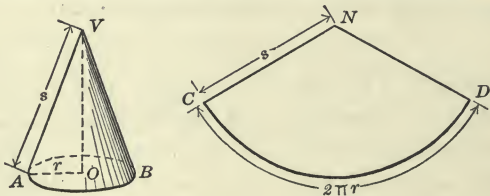
whose arc is equal to the circumference of the base of the cone. Then, by § 508, *the lateral area of a right circular cone is equal to half the product of the slant height and the circumference of the base.* $S = \frac{1}{2} sc = \pi r s.$

718. Geometric Treatment. A method for measuring the surface of a cone, which is accurate and geometric follows from a consideration similar to that of § 649. In this manner the truth of the following statement is established.

The lateral area of a right circular cone is the common limit of the areas of inscribed and circumscribed regular pyramids, as the number of faces is indefinitely doubled.

719. Theorem. *The lateral area of a right circular cone is equal to half the product of the circumference of the base and the slant height.* $S = \frac{1}{2}cs = \pi rs.$

Given a right circular cone.

To prove $S = \frac{1}{2}cs = \pi rs$, where S denotes lateral area, s slant height, c circumference of base, and r radius.

Proof. Circumscribe a regular pyramid about the cone, and let S' denote its lateral area and p the perimeter of its base.
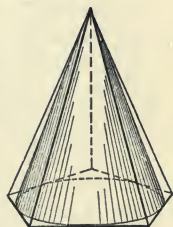
By doubling indefinitely the number of faces of the pyramid,

$$S' \to S, \text{ and } p \to c. \qquad \text{§§ 718, 488}$$

Then $\qquad\qquad \frac{1}{2}ps \to \frac{1}{2}cs. \qquad\qquad\qquad \text{§ 485 (2)}$

But $\quad S' = \frac{1}{2}ps$, being variables that are always equal. § 711

$$\therefore S = \frac{1}{2}cs. \qquad\qquad\qquad \text{§ 485 (1)}$$

Also $\qquad\qquad\qquad c = 2\pi r. \qquad\qquad\qquad\qquad \text{§ 495}$

$$\therefore S = \pi rs. \qquad\qquad\qquad\quad \text{§ 111}$$

Remark. It is to be noted that the formulas of this article apply *only* to right circular cones.

720. Theorem. *The total area T of a right circular cone is given by the formula* $T = \pi rs + \pi r^2 = \pi r(s + r).$

EXERCISES

1. Form a cone from a semicircle having a radius of 3 in. Find the lateral area of the cone. The total area.

2. Show how to cut a pattern for a tin cone that is to have the diameter of its base 4 in. and its slant height 10 in.

3. In a right circular cone show that $s = \dfrac{2S}{c} = \dfrac{S}{\pi r}$, $c = \dfrac{2S}{s}$, and $r = \dfrac{S}{\pi s}$.

4. Solve $T = \pi rs + \pi r^2$ and find $r = \dfrac{-\pi s + \sqrt{\pi^2 s^2 + 4\pi T}}{2\pi}$, and $s = \dfrac{T - \pi r^2}{\pi r} = \dfrac{T}{\pi r} - r$.

In a right circular cone, using S for lateral area, T for total area, s for slant height, r for radius, c for circumference, and h for altitude, solve the following:

5. Given $r = 4$ in., $s = 6$ in.; find S and T.

6. Given $c = 25$ in., $s = 8$ in.; find S and T.

7. Given $s = 8$ ft. 6 in., $c = 3$ ft. 7 in.; find S.

8. Given $s = 7.2$ in., $r = 5.3$ in.; find S and T.

9. Given $h = 12$ in., $r = 5$ in.; find S and T.

10. Given $c = 30$ in., $s = 8$ in.; find h and S.

11. Given $S = 200$ sq. in., $r = 6$ in.; find s.

12. Given $S = 256$ sq. cm., $s = 16$ cm.; find r.

13. Given $T = 500$ sq. cm., $r = 10$ cm.; find s.

14. The circumference of the base of a conical church steeple is 35 ft. and the altitude is 73 ft. Find the lateral area.   *Ans.* $1281-$ sq. ft.

15. Find the lateral edge and the lateral area of a regular pyramid each side of whose triangular base is 10 ft., and whose altitude is 18 ft.
          Ans. $18.90+$ ft., $273.45-$ sq. ft.

16. A tower with a regular hexagonal base has dimensions as shown in the figure. The pitch of the pyramdial roof is $1\frac{1}{2}$, which means that $OV = 1\frac{1}{2} QR$. Find the lateral area of the tower.

17. Cut out of heavy paper a sector of a circle, with a radius of 3 in. and the central angle 120°. Bring the edges together and paste them. Find the lateral area and the total area of the cone thus formed.

18. The cone formed by revolving an equilateral triangle about one of its altitudes has a lateral area equal to twice the area of the base.

19. Find the area of the surface of the solid formed by revolving an equilateral triangle, having a side of 12 in., about one of its sides.

20. Find the area of the surface of the solid formed by revolving a right triangle, having a base of 8 in. and an altitude of 6 in., about its hypotenuse.
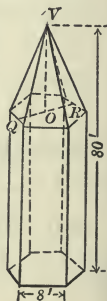
AREA OF FRUSTUM OF PYRAMID OR CONE

721. Truncated Pyramid, or Cone. The portion of a pyramid, or cone, included between the base and a section not parallel to the base is called a **truncated pyramid,** or **cone.**

722. Frustum. The portion of a pyramid, or cone, included between the base and a section parallel to the base is called a **frustum of the pyramid,** or **cone.**
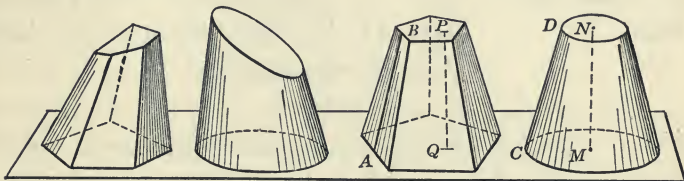
723. The base and the parallel section are called the **bases** of the frustum. The perpendicular distance between the bases is the **altitude** of the frustum.

724. The **slant height** of a frustum of a regular pyramid, or of a cone, is that portion of the slant height of the pyramid, or cone, included between the bases of the frustum.

Thus AB and CD are slant heights, and QP and MN are altitudes.

725. Midsection. If a solid has parallel bases, the section parallel to the bases and half way between them is called the **midsection.**

In the figure, NR is the midsection.

726. Prove the following facts concerning frustums:

(1) *The lateral faces of a frustum of a regular pyramid are congruent isosceles trapezoids.*

(2) *The lateral edges of a frustum of a regular pyramid are equal; and the slant height is the same for all the faces.*
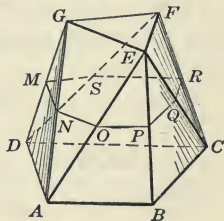
727. Theorem. *The lateral area of a frustum of a regular pyramid is equal to half the product of the slant height and the sum of the perimeters of the bases.* $S = \frac{1}{2}s(P+p)$.

Given the frustum of a regular pyramid with bases AC and EG.

To prove that $S = \frac{1}{2}s(P+p)$, where S denotes lateral area, s slant height, and P and p the perimeter of the lower and upper base respectively.
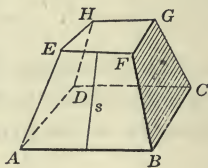
Proof.

Area $AF = \frac{1}{2}s(AB+EF)$.		§ 370
Area $BG = \frac{1}{2}s(BC+FG)$.		
etc.		

Hence $S = \frac{1}{2}s[(AB+BC+\cdots)+(EF+FG+\cdots)]$.   § 105

But $AB+BC+\cdots = P$, and $EF+FG+\cdots = p$.   Why?

$\therefore S = \frac{1}{2}s(P+p)$.   Why?

728. Theorem. *The lateral area of a frustum of a right circular cone is equal to half the product of the slant height and the sum of the circumferences of the bases.*

Given the frustum of a right circular cone.

To prove $S = \pi(R+r)s$, where S denotes lateral area, s slant height, and R and r the radius of the lower and upper base respectively.

Proof. Let m be the slant height of the cone and n the slant height of the part above the frustum.
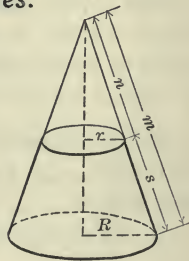
Then $S = m\pi R - n\pi r = \pi(mR - nr)$.   Why?

But $R : r = m : n$.   § 428

And therefore $mr - nR = 0$.   § 398

Hence $S = \pi(mR - nr + mr - nR)$ Why?

$= \pi(R+r)(m-n)$.   Why?

But $m - n = s$.

$\therefore S = \pi(R+r)s$.   Why?

729. Theorem. *The lateral area of a frustum of a right circular cone is equal to the product of the altitude and the circumference of a circle whose radius is the perpendicular at the midpoint of an element, and terminated by the axis.*

Outline of proof. Let a be the length of EF, the perpendicular at the midpoint of an element and terminated by the axis. Let r' be the radius of the midsection, and s and h the slant height and altitude respectively.
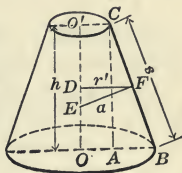
Show that $\triangle DEF \backsim \triangle ABC$.

Then $s : h = a : r'$, and $sr' = ah$.

Show that $R + r = 2r'$, where R and r are the radii of the bases.

Then, substituting in the formula of § 728, $S = 2\pi ah$.

EXERCISES

1. Prove the theorem of § 728 by considering the lateral area of a frustum of a right circular cone as the limit of the area of a frustum of a pyramid.

2. Prove the theorem of § 727 by using a method similar to that of § 728 for the frustum of a cone.

3. Show that the lateral area of a frustum of a right circular cone is equal to the slant height times the perimeter of the midsection.

4. Find the lateral area of the frustum of a regular quadrangular pyramid, if an edge of the lower base is 16 in., an edge of the upper base 12 in., and the slant height 18 in.

5. Find the total area of the frustum of a right circular cone, if the radii of the bases are 6 ft. and 8 ft. respectively, and the slant height is 7 ft.

6. A portion of the roof of a tower is a frustum of a right circular cone. The radii of the bases are 10 ft. and 6 ft. respectively, and the altitude is 8 ft. Find the number of square feet in this part of the roof.

7. The altitude of a right circular cone is h. How far from the vertex must a plane be passed parallel to the base so that the lateral area of the cone cut off shall equal the lateral area of the frustum. *Ans.* $\frac{1}{2}h\sqrt{2}$.

8. The diameter of the bottom of a pail is 10 in. and that of the top 12 in. Find the number of square inches of tin in the pail if the slant height is 11 in. *Ans.* 458.67.

9. Determine the diameter of the blank to make a pressed basin of the form shown in the figure. The depth is 3 in., the bottom has a diameter of 6 in., the top an inside diameter of 7 in., and the rim is $\frac{1}{2}$ in. wide.
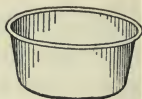
Ans.   11.4+ in.

SUGGESTION.   Consider the rim as a ring between two concentric circles. The blank must have an area equal to the total area of the basin.

10. A pie tin has a diameter of $7\frac{1}{3}$ in. at the bottom and 9 in. at the top inside. It is 1 in. deep and has a flange $\frac{3}{8}$ in. wide around the edge. Find the diameter of the blank required to make the tin.

11. Find the area generated by revolving a square a inches on a side about a diagonal.

12. A tower whose cross section is a regular octagon 6 ft. on a side has a roof that is full pitch. Find the area of the roof.

By full pitch is meant that the vertex of the roof is the same distance above the plates as the width of the tower from face to face.

13. The conical roof over a water tank is half pitch. Find its area if the diameter of the tank is 12 ft. and the roof projects $1\frac{1}{2}$ ft. at the eaves.

Ans.   221.5 − sq. ft.

By half pitch is meant that the vertex of the roof is one-half the diameter of the tower above the top of the tower.

14. Find the total area of the solid generated by revolving an isosceles trapezoid about a line connecting the middle points of its bases, if the bases are 8 in. and 12 in. respectively, and the angles at one base are each 60°.

15. Determine how to draw a pattern for the upper part of the funnel with dimensions as shown, the diameter at A being $\frac{1}{2}$ in. and at B, 1 in. The allowance for the seam can be added.

SUGGESTION.   The length $OP = x$ may be determined as follows:
$x : x+3 = 1 : 4.$ $\therefore x = 1.$

The radius to use is therefore 4 in.

To determine the central angle u, $\dfrac{\widehat{QnR}}{8\pi} = \dfrac{u}{360°}.$

But $\widehat{QnR} = 4\pi.$ $\therefore u = 180°.$

16. Determine how to draw a pattern for the lower part of the funnel shown in the figure of the previous exercise.
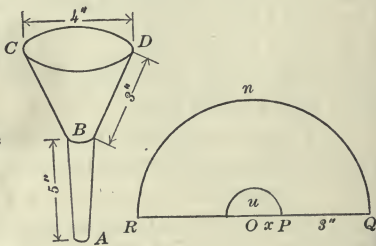
VOLUMES OF PYRAMIDS AND CONES

730. Inscribed Prisms. If the altitude of a triangular pyramid is divided into equal parts by a series of planes parallel to the base, the prisms within the pyramid and having the sections formed by the parallel planes as bases, and each having one of the segments of one of the lateral edges of the pyramid as one of its lateral edges, are called a **set of inscribed prisms.**

In the figure M, N, and R are a set of inscribed prisms, each being entirely within the pyramid.
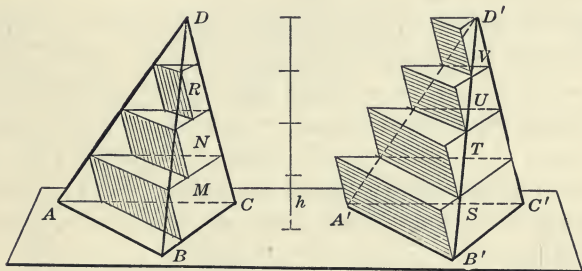
731. Circumscribed Prisms. In a similar manner, the prisms having the base of the pyramid and the sections as bases are called a **set of circumscribed prisms.**

In the figure, S, T, U, and V are a set of circumscribed prisms, each being partly outside of the pyramid.

732. The two sets of such prisms, formed by using the same lateral edge and the same parallel planes, are called **corresponding sets of inscribed and circumscribed prisms.**

733. The following statement may be considered as evident from the preceding:

The volume of a triangular pyramid is definite, and is greater than that of any set of inscribed prisms, and less than that of any set of circumscribed prisms.

734. Theorem. *The difference in volume between a set of inscribed prisms and the corresponding set of circumscribed prisms is the volume of the prism upon the base of the pyramid.*

735. Theorem. *The volume of a triangular pyramid is the common limit of the volumes of the set of inscribed prisms and the corresponding set of circumscribed prisms, as the number of these prisms is increased indefinitely.*

Given the triangular pyramid *D-ABC*, a set of inscribed prisms, and corresponding set of circumscribed prisms.

To prove that volume of *D-ABC* is the common limit of these two corresponding sets of prisms.

Outline of proof. The volume of the triangular pyramid is greater than the sum of the inscribed prisms and less than the sum of the circumscribed prisms.   § 733

Therefore, the volume of the pyramid differs from either by an amount less than the volume of the prism upon the base of the pyramid.   § Why?

But the volume of this prism can be made as small as desired, that is, it has zero for a limit.   § 485 (2)

Hence, the volume of the set of inscribed prisms and the set of circumscribed prisms, each have the volume of the triangular pyramid for a limit.   § 483
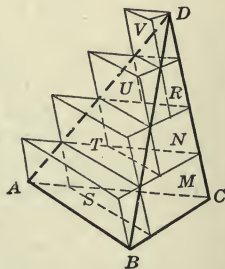
EXERCISES

1. Corresponding sets of inscribed and circumscribed prisms are formed in a triangular pyramid whose base has an area of 25 sq. in. Compute the series of differences between the corresponding sets when their altitudes are successively 0.1 in., 0.01 in., 0.001 in., and 0.0001 in.

2. The base of a pyramid is an equilateral triangle 3 in. on a side and its altitude is 7 in. Find the difference between the corresponding sets of inscribed and circumscribed prisms when the altitude of each prism is $\frac{1}{1000}$ of the altitude of the pyramid.

3. Could a quadrangular pyramid be used instead of a triangular pyramid in the discussion of §§ 730-733?

736. Theorem. *The volumes of two triangular pyramids, having equivalent bases and equal altitudes, are equivalent.*
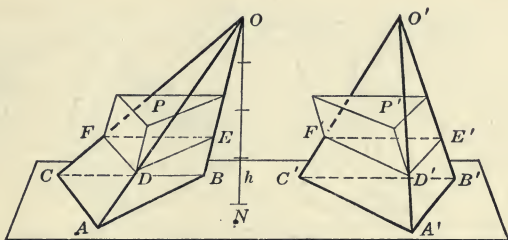
Given the triangular pyramids $O\text{-}ABC$ and $O'\text{-}A'B'C'$, having equivalent bases and equal altitudes; and, for convenience, having their bases in the same plane.

To prove that $V = V'$, where V denotes the volume of $O\text{-}ABC$ and V' the volume of $O'\text{-}A'B'C'$.

Proof. Let the common altitude ON be divided into any number of equal parts, and pass planes through the points of division parallel to the plane of the bases. Construct the set of circumscribed prisms for each pyramid upon the sections formed by these planes.

Then the sections of the pyramids made by each plane are equivalent. § 710

Let DEF and $D'E'F'$ be the sections made by any one of these planes, and P and P' the corresponding prisms.

Then volume of $P =$ volume of P'. § 686

Let S denote the volume of the sum of the prisms in the set of which P is one; and S' that of which P' is one.

Then $S = S'$.   § 105

But $S \rightarrow V$, and $S' \rightarrow V'$.   § 735

And S and S' are variables that are equal for all their successive values as the number of the divisions of ON is increased indefinitely.

$$\therefore V = V'. \qquad \text{§ 485 (1)}$$

737. Theorem. *The volume of a triangular pyramid is equal to one-third the product of its base and altitude.* $V = \frac{1}{3}Bh$.
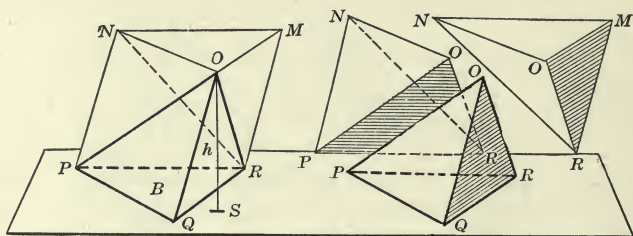
Given the triangular pyramid $O\text{-}PQR$.

To prove that $V = \frac{1}{3}Bh$, where V denotes the volume of the pyramid, B the area of its base, and h its altitude.

Proof. Upon the base PQR construct a triangular prism NR of altitude h, and having its lateral edges parallel to OQ.

The prism NR is divided into three triangular pyramids by the sections OPR and ONR.

Pyramid $R\text{-}NOM = $ pyramid $O\text{-}PQR$.   § 736
Pyramid $R\text{-}OPQ = $ pyramid $R\text{-}ONP$.   § 736

But pyramid $R\text{-}OPQ$ is the same as pyramid $O\text{-}PQR$.

Therefore the three triangular pyramids are equal, and $O\text{-}PQR$ is one-third the volume of prism NR.

But volume of prism $NR = Bh$.   § 684
$$\therefore V = \frac{1}{3}Bh. \qquad \text{Why?}$$

738. Theorem. *The volume of any pyramid is equal to one-third the product of its base and altitude.* $V = \frac{1}{3}Bh$.

Suggestion. From one vertex of the base draw all the diagonals of the base. Pass planes through the vertex of the pyramid and each of these diagonals. All the pyramids have the same altitude. Apply § 737 to each triangular pyramid formed and add the results.
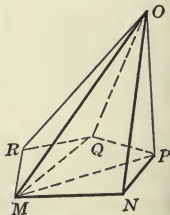
739. By a like treatment to that referred to in § 650, the truth of the following statement may be established:

The volume of a circular cone is the common limit of the volumes of inscribed and circumscribed pyramids with regular polygons for bases, as the number of faces is indefinitely doubled.

740. Theorem. *The volume of a circular cone is equal to one-third the product of its base and altitude.* $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h.$

Given a circular cone.

To prove $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$, where V denotes volume, B area of base, h altitude, and r radius.

Proof. Inscribe a pyramid with a regular polygon for base, and let V' denote its volume and B' the area of its base.
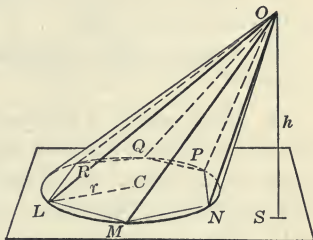
By doubling indefinitely the number of faces of the pyramid,

$$V' \rightarrow V, \text{ and } B' \rightarrow B. \qquad \text{§ 739, § 489}$$
$$\frac{1}{3}B'h \rightarrow \frac{1}{3}Bh. \qquad \text{§ 485 (2)}$$

But $V' = \frac{1}{3}B'h$, being variables that are always equal. § 738

$$\therefore V = \frac{1}{3}Bh. \qquad \text{§ 485 (1)}$$

Also $B = \pi r^2.$ § 498

$$\therefore V = \frac{1}{3}\pi r^2 h. \qquad \text{§ 111}$$

Remark. It is to be noted that the formulas of this article apply to *any* circular cone. Compare with the remark of § 719.

EXERCISES

1. Show that for any pyramid or circular cone $B = \dfrac{3V}{h}$, and $h = \dfrac{3V}{B}$.

2. Find the volume of a square pyramid whose base is 16 in. on a side, and whose altitude is 10 in.

3. Find the volume of a pyramid whose base is a regular hexagon 8 in. on a side, and whose altitude is 14 in.

4. Find the volume of a pyramid whose base is a square $3\sqrt{2}$ in. on a side, and whose altitude is $6\frac{3}{4}$ in.

5. The base edges of a triangular pyramid are 6 in., 8 in., and 10 in. Find its volume if the altitude is 15 in.

6. Find the volume of a circular cone whose radius is 5 in. and altitude 10 in.

7. Find the volume of a circular cone whose altitude is 70 ft. and the circumference of whose base is 31 ft.

8. Derive a formula for finding the radius of a circular cone in terms of the volume and altitude.

9. Derive a formula for finding the altitude of a circular cone in terms of the volume and radius.

10. A right circular cone has a slant height s and a radius r; find its volume V. $\qquad$ *Ans.* $V = \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$.

11. The Pyramid of Cheops has a square base 720 ft. on a side, and an altitude of 480 ft. Find the number of cubic yards in it.

12. What is the locus of the vertices of all pyramids having the same base and equal volumes?

13. Find the total area and the volume of the solid generated by revolving an equilateral traingle of side a about one side. Find the values if $a = 2$ in. $\qquad$ *Ans.* Area $= a^2\pi\sqrt{3}$; 21.765 sq. in.
Volume $= \frac{1}{4}a^3\pi$; 6.283 cu. in.

14. Find the total area and the volume of the solid generated by revolving a parallelogram with sides 22 in. and 16 in., and larger angle 120°, about one of the longer sides. *Ans.* 3308.3 − sq. in.; 13270+ cu. in.

15. A pyramid whose base is a square 4 ft. on a side, and whose altitude is 12 ft. is bisected by a plane parallel to the base. How far from the vertex is the plane?

16. A circular sheet of copper 3 ft. in diameter is cut in half, and each half formed into a cone. These are placed base to base as shown in the figure. This is to be used as a float. Find the number of pounds it will support if the sheet copper weighs 0.92 lb. per square foot and water weighs 62.5 per cubic foot.
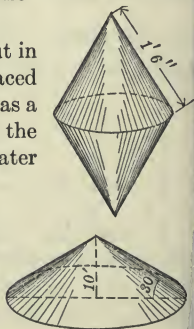
17. Hard coal dumped in a pile lies at an angle of 30° with the horizontal. Estimate the number of tons in a pile in the shape of a right circular cone having an altitude of 10 ft. Large egg size weighs 38 lb. per cubic foot. *Ans.* About 60 tons.

18. Show how to construct a triangular pyramid equivalent to any given pyramid having a polygon as base

741. Theorem. *The volume V of a frustum of a pyramid, or circular cone, of bases B_1 and B_2 and altitude h, is given by the formula* $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.
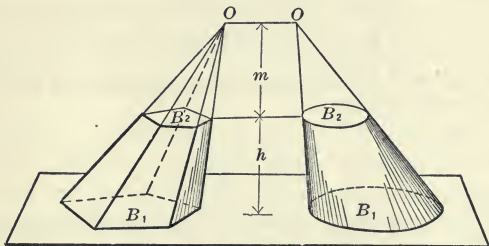
Given the frustum of a pyramid, or of a circular cone.

To prove $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$, where V denotes volume, h altitude, and B_1 and B_2 the areas of the lower base and upper base, respectively.

Proof. Complete the pyramid, or cone, and let m denote the altitude of the portion above the frustum in each case, and V_2 the volume of that portion. Also let V_1 denote the volume of the entire pyramid, or cone.

Then
$$V = V_1 - V_2.$$
$$= \frac{1}{3}B_1(h+m) - \frac{1}{3}B_2 m. \qquad \S\,740$$
$$= \frac{1}{3}[B_1 h + (B_1 - B_2)m].$$

It remains to eliminate m. To do this one more equation is needed. This is given by

$$\frac{B_1}{B_2} = \frac{(h+m)^2}{m^2}. \qquad \S\,709$$

$$\frac{\sqrt{B_1}}{\sqrt{B_2}} = \frac{h+m}{m}.$$

Solving for m,
$$m = \frac{h\sqrt{B_2}}{\sqrt{B_1} - \sqrt{B_2}}.$$

Substituting this for m,
$$V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2}).$$

742. Theorem. *The volume V of a frustum of a circular cone having bases with diameters d_1 and d_2, and radii r_1 and r_2, and an altitude h, is given by the formulas:*

$$(1)\quad V = \tfrac{1}{3}\pi h(r_1{}^2 + r_2{}^2 + r_1 r_2),$$
$$(2)\quad V = \tfrac{1}{12}\pi h(d_1{}^2 + d_2{}^2 + d_1 d_2).$$

Substitute $\pi r_1{}^2$ and $\pi r_2{}^2$ for B_1 and B_2 respectively of § 741 to find (1). Use $\tfrac{1}{4}\pi d_1{}^2$ and $\tfrac{1}{4}\pi d_2{}^2$ for (2).

EXERCISES

1. Find the volume of a frustum of a pyramid with square bases having edges of 9 ft. and 4 ft. respectively, and altitude 12 ft.

2. The radii of the bases of a frustum of a right circular cone are respectively 10 cm. and 12 cm. Find its volume if the altitude is 16 cm. Find its lateral area.

3. Find the volume of a frustum of a regular hexagonal pyramid the bases being respectively 8 in. and 6 in. on an edge, and the altitude 10 in.

4. The frustum of a cone has an altitude of 8 in. and the radii of its bases are 7 in. and 4 in. respectively. Find its volume. Does the frustum have to be cut from a right cone in order that its volume can be found?

5. Derive a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the radii of the bases.

6. The diameter of the top of a water pail is 12 in., the diameter of the bottom 10 in., and the altitude $10\tfrac{1}{2}$ in. How many quarts will the pail hold? One quart is 57.75 cu. in. *Ans.* 17.33−.

7. Find the weight of an iron casting in the form of a right circular cone of diameter 8 in. and altitude 12 in., with a circular hole having a diameter of 2 in. through the center. (Cast iron weighs 0.26 lb. per cubic inch.) *Ans.* 44.1 lb.

8. A tank is made in the form of an inverted frustum of a cone. The slant height is 14 ft. and makes an angle of 60° with the horizontal, and the lower base is a circle 20 ft. in diameter. Find the volume of the tank in barrels. Use 1 bbl. = 4.211 cu. ft. *Ans.* 1685.4 bbl.

9. A tank is in the form of an inverted cone with its vertex angle 60° and its axis vertical. Find the depth of water in the cone if it has been running in for 10 minutes at the rate of 8 cu. ft. per minute.
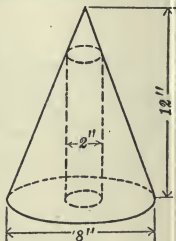
SIMILAR PYRAMIDS AND CONES

743. Similar Pyramids. Pyramids that are formed from the same pyramidal surface by parallel planes are **similar**.

744. Similar Cones. Cones that are formed from the same conical surface by parallel planes are **similar**.

745. Theorem. *The lateral areas, or the total areas, of two similar regular pyramids, or right circular cones, are to each other as the squares of two corresponding lines, where the corresponding lines may be edges, elements, altitudes, slant heights or radii.*

One case only will be proved. The other cases are left as exercises.

Given two similar regular pyramids O-$ACDEF$ and O-$A'C'D'E'F'$.

To prove $\dfrac{S}{S'} = \dfrac{h^2}{h'^2}$, and $\dfrac{T}{T'} = \dfrac{h^2}{h'^2}$, where S and S' denote the lateral areas of O-$ACDEF$ and O-$A'C'D'E'F'$ respectively, h and h' altitudes, and T and T' total areas.
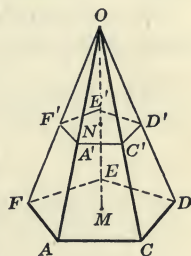
Proof. If p and p' are the perimeters of the respective bases, and s and s' the slant heights, $S = \frac{1}{2}ps$, and $S' = \frac{1}{2}p's'$.   § 711

Then $\dfrac{S}{S'} = \dfrac{\frac{1}{2}ps}{\frac{1}{2}p's'} = \dfrac{p}{p'} \cdot \dfrac{s}{s'}.$ Why?

$\therefore \dfrac{S}{S'} = \dfrac{h^2}{h'^2}$ since $\dfrac{p}{p'} = \dfrac{h}{h'}$, and $\dfrac{s}{s'} = \dfrac{h}{h'}.$ Why?

Further, if B and B' are the areas of the respective bases,
$\dfrac{B}{B'} = \dfrac{h^2}{h'^2}$ and so $\dfrac{S}{S'} = \dfrac{B}{B'}$, and $\dfrac{S}{B} = \dfrac{S'}{B'}.$ Why?

Then $\dfrac{S+B}{B} = \dfrac{S'+B'}{B'}$, and $\dfrac{S+B}{S'+B'} = \dfrac{B}{B'}.$ §§ 406, 405

$\therefore \dfrac{T}{T'} = \dfrac{h^2}{h'^2}$ since $S+B = T$, and $S'+B' = T'.$ Why?

746. Theorem. *The volumes of two similar pyramids, o circular cones, are to each other as the cubes of two correspondin lines; where the corresponding lines may be edges, elements, alti tudes, slant heights, or radii.*

Using V and V' for the respective volumes, B and B' for bases and h and h' for altitudes,

$$\frac{V}{V'} = \frac{\frac{1}{3}Bh}{\frac{1}{3}B'h'} = \frac{B}{B'} \cdot \frac{h}{h'}.$$

But $\dfrac{B}{B'} = \dfrac{h^2}{h'^2}$.   Why? $\therefore \dfrac{V}{V'} = \dfrac{h^3}{h'^3}$.   Why

747. Definition. The **vertex angle** of a right circular con is twice the angle between an element and the axis.

748. Theorem. *All right circular cones with equal verte angles are similar.*

EXERCISES

1. Two cones generated by similar right triangles revolving abou corresponding sides are similar.

2. Two cones are generated by revolving two similar right triangle about corresponding sides of length 7 in. and 10 in. respectively. Fin the ratio of the lateral areas of the cones. The ratio of the total area The ratio of the volumes.

3. The total area of a right circular cone is 300 sq. in. and its altitud is 8 in. Find the total area of the cone cut off by a plane parallel to th base and 6 in. from the vertex.

4. How far from the vertex of a right circular cone, or a regular pyra mid, of altitude h must a plane be passed parallel to the base so that th lateral area of the part cut off shall be one-half that of the original?

5. From a given pyramid, cut off a frustum whose volume shall b $\frac{26}{27}$ that of the given pyramid. Must the pyramid be regular?

6. The volumes of two pyramids are 343 cu. in. and 1000 cu. in. respec tively. The lateral area of the smaller is 100 sq. in. Find the latera area of the larger.

7. The lateral areas of two similar right circular cones are in the ratic of 9 : 64. Find the volume of the smaller if the volume of the larger i 1024 cu. in.

QUESTIONS

1. What is a pyramidal surface? A conical surface? Is the directrix necessarily closed?

2. What are the formulas or rules for finding the areas of pyramids, cones, and frustums of pyramids and cones?

3. What are the formulas or rules for finding the volumes of each of these solids?

4. Can you find the area of an oblique circular cone? Of an oblique pyramid? Can you find the volume of each of these?

5. If the altitude of a cone is divided into 10 equal parts by planes passed parallel to the base, how does the area of each section of the cone formed by these planes compare with the base of the cone?

6. Trace the steps in finding the volume of a triangular pyramid. Of any pyramid. Of a cone.

7. What is the relation between two parallel sections of a pyramid or cone?

8. State theorems concerning the sections of a cone. Of a pyramid.

9. What effect does it have upon the volume of a pyramid or cone if the area of the base is doubled? If the altitude is doubled? If both base and altitude are doubled?

10. What effect does it have upon the lateral area of a right circular cone if the circumference of the base is doubled? If the area of the base is doubled?

11. Point out forms of this chapter that occur in nature. That occur in the arts.

GENERAL EXERCISES

1. Given the following formulas for the right circular cone: (1) $V = \frac{1}{3}\pi r^2 h$, (2) $S = \pi r s$, (3) $S = \pi r \sqrt{h^2 + r^2}$, (4) $T = A + \pi r \sqrt{h^2 + r^2}$, (5) $T = \pi r^2 + \pi r \sqrt{h^2 + r^2}$; solve each for each letter appearing.

2. In a right circular cone, $A = 50\pi$, and $V = 200\pi$. Find r, h, and s.

3. If R and r are the radii of the two bases of a frustum of a cone, show that $R + r = \sqrt{\dfrac{3V}{\pi h} + Rr}$.

4. Find the volume and the total area of a right circular cone whose radius of base is 6 in. and altitude is 5.3 in. *Ans.* 199.8 cu. in.; 263.9 sq. in.

5. Find the weight of a tapered brick smoke stack 175 ft. high in the form of a frustum of a right circular cone enclosing a cylinder. The inner diameter is 10 ft., the wall is 4 ft. thick at the base and 1 ft. 6 in. at the top. One cubic foot of brick weighs 112 pounds.   *Ans.* 1095.5 tons.

6. Find the lateral edge, lateral area, and volume of a regular pyramid each side of whose triangular base is 5 ft., and whose altitude is 9 ft.
Ans. 9.45 ft.; 68.36 sq. ft.; 32.5 cu. ft.

7. A bin in a warehouse is 12 ft. square. A hopper is constructed on the base, which has a slope of 1 : 1. The distance from the vertex of the hopper to the top of the bin is 18 ft. Find the capacity of the bin if 1 bushel equals $\frac{5}{4}$ cu. ft.

8. The largest possible cylinder of diameter 6 in. is cut from a right circular cone having a diameter and altitude of 10 in. and 26 in. respectively. Find the volume of the cylinder.

9. Find the area and volume of the solid generated by revolving an isosceles triangle of base 10 in. and equal sides 16 in., about its base. Find the area and the volume of the solid when this triangle is revolved about one of its equal sides.

10. In the frustum of a pyramid whose base is 50 sq. ft. and altitude 6 ft., the basal edge is to the corresponding top edge as 5 to 3. Find the volume of the frustum.   *Ans.* 196 cu. ft.

11. The base of a pyramid is a rectangle 10 in. by 8 in. Find the volume if each lateral edge makes with the base an angle of 45°.

12. The total surface of a regular quadrangular pyramid is T, and its eight edges are equal. Find the length of an edge.

13. Find the volume of a cone of revolution whose slant height is equal to the diameter of its base, and whose total area is T.

14. Show how to cut a pattern for the frustum of a right circular cone, if the upper and lower bases have diameters of 4 in. and 6 in., respectively, and the altitude is 8 in.

15. A right circular cone whose altitude is 8 in. and radius 6 in. rolls on a floor without slipping, making one complete revolution. What is the shape of the surface covered? Find its area.

16. A pyramid whose altitude is 12 in. weighs 30 pounds. At what distance from the vertex must it be cut by a plane parallel to its base so that the frustum cut off shall weigh 20 pounds?

17. A cube is cut by a plane passed through the other extremities of the three edges meeting at a vertex. What part of the volume of the cube is thus cut off?

CHAPTER IX

PRISMATOIDS AND POLYEDRONS

PRISMATOIDS

749. A polyedron all of whose vertices lie in two parallel planes is called a **prismatoid.** In either plane, the vertices may lie in any polygon, as Fig. (1); or, in one plane, the vertices may lie in a line, as in Fig. (2); or there may be a point only, as in a pyramid. The faces whose vertices are in the
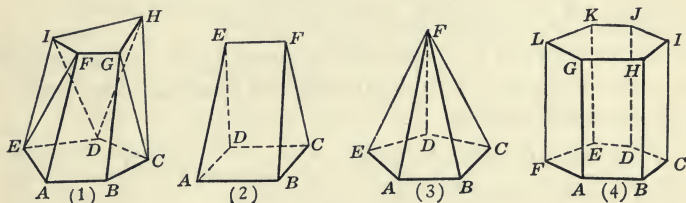
parallel planes are evidently triangles, or quadrilaterals of types that have at least two parallel sides. Why?

750. The **altitude** of a prismatoid is the distance between the two parallel planes.

EXERCISES

1. Show that a prism is a prismatoid whose bases are congruent polygons.

2. Show that a frustum of a pyramid is a prismatoid whose bases are similar polygons.

3. Draw or describe a prismatoid that has a face that is a trapezoid. A parallelogram. A rectangle.

4. Compute the area of the midsection of a pyramid whose base is a regular hexagon having a side of 6 in.

5. Compute the areas of the midsection of the prismatoid in Fig. (2) if $ABCD$ is a rectangle having $AB = 18$ in. and $BC = 11$ in., and $EF = 12$ in.

751. Theorem. *If B_1 and B_2 are the areas of the two bases, M the area of the midsection, and h the altitude, then the volume V of a prismatoid is given by the formula $V = \frac{1}{6}h(B_1 + B_2 + 4M)$.*

Given a prismatoid of volume V, altitude h, bases B_1 and B_2, and midsection M.

To prove $V = \frac{1}{6}h(B_1 + B_2 + 4M)$.

Proof. If any lateral face is not a triangle divide the face into triangles by a diagonal.

Take any point O in the midsection and connect it with the vertices of the prismatoid and the midsection. These lines form triangles with the edges of the prismatoid, which separate the prismatoid into pyramids that have the bases and faces of the prismatoid as bases.

(1) Pyramid $O\text{-}ABCD = \frac{1}{6}hB_1$.   § 737

(2) Pyramid $O\text{-}NPQ = \frac{1}{6}hB_2$.

Each of the pyramids like $O\text{-}CDP$, which may be called a lateral pyramid, can have its volume expressed in terms of the part of the midsection common to it. To prove this consider $O\text{-}CDP$ apart from the prismatoid, and draw HD.

Altitude of each of pyramids $P\text{-}OHI$ and $D\text{-}OHI$ is $\frac{1}{2}h$.

Pyramid $P\text{-}OHI = \frac{1}{6}h$ times area OHI.   § 737

Pyramid $D\text{-}OHI = \frac{1}{6}h$ times area OHI.   Why?

Pyramid $O\text{-}CDH = 2$ times pyramid $O\text{-}HDI$.   Why?

Then pyramid $O\text{-}CDH = \frac{2}{6}h$ times area OHI.

Hence pyramid $O\text{-}CDP = \frac{4}{6}h$ times area OHI.   Why?

Similarly each lateral pyramid is equal to $\frac{4}{6}h$ times the area of its part of the midsection.

(3) Therefore the sum of the lateral pyramids $= \frac{4}{6}hM = \frac{1}{6}h \cdot 4M$.   Why?

Adding (1), (2), and (3),

$$V = \frac{1}{6}h(B_1 + B_2 + 4M).$$

752. Other Applications of the Prismatoid Formula. The formula for the volume of a prismatoid can be applied also to various other solids such as cylinders, cones, spheres, combinations of these, and various other forms that cannot well be described here.

The prismatoid formula, because of its wide application, is used often by engineers and others in determining the volumes of embankments, abutments, and other irregular solids of various kinds.

EXERCISES

1. Derive the following formulas from the formula for the volume of a prismatoid:

(1) Prism and cylinder, $V = Bh$.

(2) Pyramid and cone, $V = \frac{1}{3}Bh$.

(3) Frustum of pyramid or cone, $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

Derivation for frustum.

Given $V = \frac{1}{6}h(B_1 + B_2 + 4M)$.

To derive $V = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

It is then necessary to express M in terms of B_1 and B_2.

Using the notation of § 741, $\dfrac{B_1}{M} = \dfrac{(h+m)^2}{(\frac{1}{2}h+m)^2}$, and $\dfrac{B_2}{M} = \dfrac{m^2}{(\frac{1}{2}h+m)^2}$.   § 709

Or $\dfrac{\sqrt{B_1}}{\sqrt{M}} = \dfrac{h+m}{\frac{1}{2}h+m}$, and $\dfrac{\sqrt{B_2}}{\sqrt{M}} = \dfrac{m}{\frac{1}{2}h+m}$.

Solving each of these for m, $m = \dfrac{h(\sqrt{M} - \frac{1}{2}\sqrt{B_1})}{\sqrt{B_1} - \sqrt{M}}$ from the first;

and $m = \dfrac{\frac{1}{2}h\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$ from the second.

Then $\dfrac{h(\sqrt{M} - \frac{1}{2}\sqrt{B_1})}{\sqrt{B_1} - \sqrt{M}} = \dfrac{\frac{1}{2}h\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$.

Dividing by $\frac{1}{2}h$, $\dfrac{2\sqrt{M} - \sqrt{B_1}}{\sqrt{B_1} - \sqrt{M}} = \dfrac{\sqrt{B_2}}{\sqrt{M} - \sqrt{B_2}}$.

Then $\dfrac{\sqrt{M}}{2\sqrt{M} - \sqrt{B_1}} = \dfrac{\sqrt{M}}{\sqrt{B_2}}$, and $2\sqrt{M} = \sqrt{B_1} + \sqrt{B_2}$.   § 406

Or $4M = B_1 + B_2 + 2\sqrt{B_1 B_2}$.

$\therefore V = \frac{1}{6}h(2B_1 + 2B_2 + 2\sqrt{B_1 B_2}) = \frac{1}{3}h(B_1 + B_2 + \sqrt{B_1 B_2})$.

2. Show that, in general, the midsection of a prismatoid has as many sides as the two bases combined. When will this not be so?

3. Find the volume of the frustum of a square pyramid, the lower base being 8 ft. square, the upper base 6 ft. square, and the altitude 12 ft. Solve both by the frustum formula and the prismatoid formula, and compare.

4. Both bases of a prismatoid of altitude h are squares, and the lateral faces isosceles triangles. The sides of the upper base are parallel to the diagonals of the lower base and half the length of these diagonals. If b is a side of the lower base, find the volume. *Ans.* $\frac{5}{6}b^2h$.

5. Find the volume of the prismatoid with dimensions as shown in the figure. The bases are right triangles having their corresponding sides parallel.
*Ans.* 3420 cu. in.
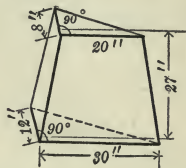
6. Use the prismatoid rule to find the volume of a frustum of a pyramid whose bases are regular hexagons 10 in. and 6 in. on a side respectively, and whose altitude is 18 in.

7. Find the volume of the frustum of the preceding exercise by § 741 and compare the result with that of the preceding.

8. The volume of a truncated triangular prism is readily determined by the prismatoid formula. To do this consider one face and the opposite edges as the bases.

Find the volume of the truncated triangular prism shown in the figure. The base ABC, which has a right angle at C, is a right section.
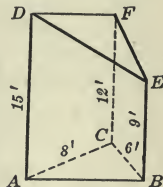
9. Show that the volume of any truncated triangular prism is equal to the product of the area of a right section and one-third the sum of the three edges.

Suggestion. Let the edges be c, d, and e, and the altitude and base of a right section a and b respectively.

Consider the prism as a prismatoid with bases $ABED$ and CF, and midsection M.

Then $V = \frac{1}{6}a\left[\frac{1}{2}b(c+d)+0+4M\right]$. § 751

But $M = \frac{1}{2}b \cdot \frac{1}{2}\left[\frac{1}{2}(c+e)+\frac{1}{2}(d+e)\right]$.

$\therefore V = \frac{1}{3}ab \cdot \frac{1}{3}(c+d+e) =$ area of right section $\times \frac{1}{3}(c+d+e)$.
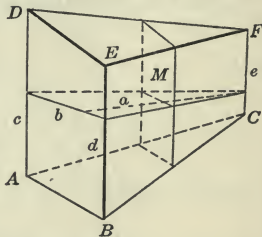
10. A truncated right triangular prism has for base an isosceles triangle whose sides are 10 in., 10 in., and 8 in. Find its volume if the three lateral edges are 5 in., 7 in., and 11 in. respectively.

POLYEDRONS

753. Regular Polyedrons. A polyedron whose faces are congruent regular polygons and whose polyedral angles are congruent is called a **regular polyedron**.

A polyedron of four faces is a **tetraedron**; one of six faces, a **hexaedron**; one of eight faces, an **octaedron**; one of twelve faces, a **dodecaedron**; and one of twenty faces, an **icosaedron**.

It is instructive to construct the regular polyedrons as shown in the following figures. Draw on cardboard the diagrams shown below, cut along the full lines, and fold along the dotted lines. The edges that meet may be fastened with gummed paper. Durable models may be constructed from tin and soldered at the edges.

There are many interesting facts connected with the five regular polyedrons. The so-called fourteenth, fifteenth, and sixteenth books of Euclid give many theorems and problems concerning these solids.
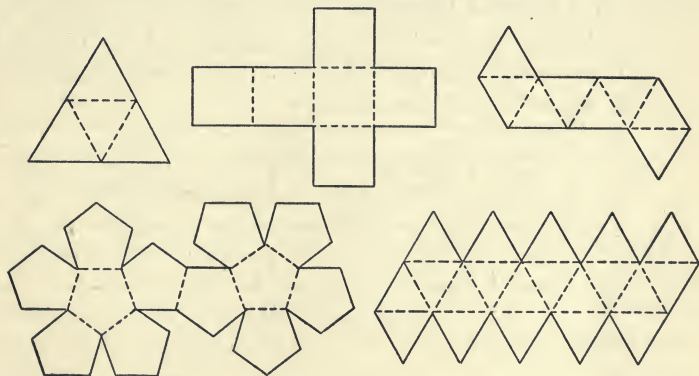
754. Theorem. *There can be no more than five regular poly-edrons.*

The proof of this theorem depends upon the facts:

(a) At least three planes must meet to form a polyedral angle. (§ 603).

(b) The sum of the face angles of a polyedral angle is less than 360°. (§ 609).

(c) The faces of a regular polyedron are congruent regular polygons.

(1) *Faces equilateral triangles.*

A polyedral angle can be formed of three, four, or five equilateral triangles, and no more.   Why?

Therefore no more than three regular polyedrons can be formed having equilateral triangles as faces.

(2) *Faces squares.*

A polyedral angle can be formed of three squares, and no more.   Why?

Therefore no more than one regular polyedron can be formed having squares for faces.

(3) *Faces regular pentagons.*

A polyedral angle can be formed of three regular pentagons, and no more.   Why?

Therefore no more than one regular polyedron can be formed having regular pentagons for faces.

Regular polygons with a greater number of sides than five cannot be used to form polyedral angles because the sum of three or more angles of such polygons is equal to or greater than 360°.

Hence there can be no more than five regular polyedrons.

755. Relation between the Number of Faces, Edges, and Vertices of a Regular Polyedron. It is interesting to note the relation between the number of faces, edges, and vertices of a regular polyedron. Complete the following table, and compare with the table on page 67.

Name	Number of Faces	Form of Faces	Number of Edges	Number of Vertices	Number of Faces at Each Vertex	Sum of Face Angles
Tetraedron	4	Equi. △	6	4	3	720°
Hexaedron						
Octaedron						
Dodecaedron						
Icosaedron						

756. Euler's Theorem. This theorem is stated because of its interest. The proof is too difficult to be given here. As an exercise it may be tested for the regular polyedrons. Use the preceding table.

Theorem. *In any polyedron the number of edges plus two is equal to the sum of the number of faces and the number of vertices.*

NOTE. Leonhard Euler was born in Switzerland in 1707 and died in Petrograd in 1783. He was one of the greatest physicists, astronomers, and mathematicians of the eighteenth century.

757. Theorem. *The sum of the face angles of any polyedron is equal to 360° multiplied by two less than the number of vertices.*

The proof of this theorem is based upon the preceding theorem. Test it for the regular polyedrons.

EXERCISES

1. Find the area of the surface of a regular tetraedron 3 in. on an edge. Of a regular octaedron 4 in. on an edge. Of a regular icosaedron 6 in. on an edge. *Ans.* 15.588+ sq. in.; 55.426 − sq. in.; 311.769+ sq. in.

2. Find the area of the surface of a regular dodecaedron 2 in. on an edge. *Ans.* 82.583 − sq. in.

3. If the edge of a regular tetraedron is e, show that its volume is given by the formula $V = \frac{1}{12}e^3\sqrt{2}$.

SOLUTION. $V = \frac{1}{3}ON \times$ area of $\triangle ABC$.
$DC = \frac{1}{2}e\sqrt{3}$. See p. 166, Ex. 7 (3).
Area of $\triangle ABC = \frac{1}{2}AB \times DC = \frac{1}{4}e^2\sqrt{3}$.
$OC = \frac{2}{3}DC = \frac{1}{3}e\sqrt{3}$. §211.
$ON = \sqrt{\overline{CN}^2 - \overline{OC}^2} = \sqrt{e^2 - (\frac{1}{3}e\sqrt{3})^2} = e\sqrt{\frac{2}{3}}$.
$\therefore V = \frac{1}{3}e\sqrt{\frac{2}{3}} \times \frac{1}{4}e^2\sqrt{3} = \frac{1}{12}e^3\sqrt{2}$.
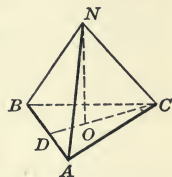
4. If the edge of a regular octaedron is e, show that its volume is given by the formula $V = \frac{1}{3}e^3\sqrt{2}$.

5. Compare the volumes of a regular tetraedron, a cube, and a regular octaedron, each 2 in. on an edge.

6. Find the area and volume of a regular tetraedron having an altitude of 8 in.

758. Theorem. *The volume of two tetraedrons that have a triedral angle of one congruent to a triedral angle of the other are to each other as the products of the three edges of these triedral angles.*
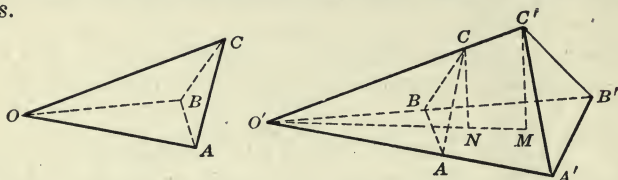

Given the tetraedrons O-ABC and O'-$A'B'C'$, with triedral $\angle O =$ triedral $\angle O'$.

To prove $\dfrac{V}{V'} = \dfrac{OA \cdot OB \cdot OC}{O'A' \cdot O'B' \cdot O'C'}$, where V and V' denote the volumes of the tetraedrons.

Proof. Place tetraedron O-ABC so that triedral $\angle O$ will coincide with triedral $\angle O'$.

Draw CN and $C'M$ perpendicular to $O'A'B'$.

CN and $C'M$ determine a plane which intersects $O'A'B'$ in $O'M$.   Why?

Now $\dfrac{V}{V'} = \dfrac{O'AB \cdot CN}{O'A'B' \cdot C'M} = \dfrac{O'AB}{O'A'B'} \cdot \dfrac{CN}{C'M}.$ Why?

But $\dfrac{O'AB}{O'A'B'} = \dfrac{O'A \cdot O'B}{O'A' \cdot O'B'}.$ § 375

Further $\triangle O'NC$ and $\triangle O'MC'$ are similar rt. $\triangle$.   Why?

Then $\dfrac{CN}{C'M} = \dfrac{O'C}{O'C'}.$ § 428

$\therefore \dfrac{V}{V'} = \dfrac{O'A \cdot O'B \cdot O'C}{O'A' \cdot O'B' \cdot O'C'} = \dfrac{OA \cdot OB \cdot OC}{O'A' \cdot O'B' \cdot O'C'}.$ § 111

Compare this theorem and the proof with § 375.

759. Theorem. *Two parallelepipeds that have a triedral angle of one congruent to a triedral angle of the other are to each other as the products of the three edges of these triedral angles.*

760. Similar Polyedrons. Two polyedrons are **similar** if they have the same number of faces similar each to each and similarly placed, and have their corresponding polyedral angles congruent. It is evident that the corresponding diedral angles are also equal.

761. Theroem. *The corresponding edges of similar polyedrons are proportional.*

Corresponding edges belonging to corresponding faces are proportional by § 444. Use the fact that each edge belongs to two faces and prove the theorem.

762. Theorem. *The surfaces of two similar polyedrons are to each other as the squares of any two corresponding edges.*

Given two similar polyedrons P and P', with corresponding faces A and A', B and B', C and C', $\cdots$. Also given a and a', b and b', c and c', $\cdots$, corresponding sides of these faces respectively.

To prove $\dfrac{A+B+C+\cdots}{A'+B'+C'+\cdots} = \dfrac{n^2}{n'^2}$, where n and n' are any two corresponding edges.

Proof. $\quad\quad \dfrac{A}{A'} = \dfrac{a^2}{a'^2}, \dfrac{B}{B'} = \dfrac{b^2}{b'^2}, \dfrac{C}{C'} = \dfrac{c^2}{c'^2}, \cdots.$ §446

But $\quad\quad\quad \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'} = \cdots = \dfrac{n}{n'}.$

And hence $\quad\quad \dfrac{a^2}{a'^2} = \dfrac{b^2}{b'^2} = \dfrac{c^2}{c'^2} = \cdots = \dfrac{n^2}{n'^2}.$ Why?

Therefore $\quad\quad \dfrac{A}{A'} = \dfrac{B}{B'} = \dfrac{C}{C'} = \cdots = \dfrac{n^2}{n'^2}.$ Why?

$\therefore \dfrac{A+B+C+\cdots}{A'+B'+C'+\cdots} = \dfrac{n^2}{n'^2}.$ §403
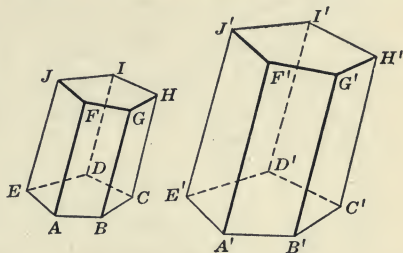
763. Theorem. *The volumes of two similar tetraedrons are to each other as the cubes of any two corresponding edges.*

Use §§ 746, 761.

764. Theorem. *The volumes of two similar polyedrons are to each other as the cubes of any two corresponding edges.*

This theorem will not be proved. It is accepted as true since it has useful applications.

EXERCISES

1. What is the ratio of the volumes of two cubes that are 5 in. and 4 in. on an edge? What is the ratio of their areas?

2. The edges of a parallelepiped are 10, 12, and 14. Find the edges of a similar parallelepiped having $\frac{1}{9}$ as great an area. Find the ratio of their volumes.

3. Two pyramids are cut from the same pyramidal surface. The lateral edges of one are 9 in., 12 in., and 14 in. and of the other 18 in., 21 in., and 20 in. Find the ratio of their volumes.

4. Find the ratio of the volume of a regular octaedron 6 in. on an edge to the volume of a regular octaedron having half as great an area.

5. The center of gravity of a tetraedron is $\frac{1}{4}$ its altitude above the base. Find the center of gravity of a regular tetraedron 8 in. on an edge.

6. The edge of a regular tetraedron is e. Find the edge of a regular tetraedron that has a volume n times as great as the volume of the given tetraedron.

7. Find the edge of a regular tetraedron such that its volume multiplied by $\sqrt{2}$ is 288.

8. Find the volume of a regular tetraedron that has an altitude of 17 in.

9. A section of a tetraedron by a plane parallel to two opposite edges is a parallelogram.

10. The midpoints of the edges of a regular tetraedron are the vertices of a regular octaedron.

11. Two similar polyedrons have volumes of 121.5 cu. in. and 4.5 cu. in., respectively. An edge of the smaller is $1\frac{1}{2}$ in. Find the corresponding edge of the larger. *Ans.* 4.5 in.

12. The area of the entire surface of a polyedron is 108 sq. in., and its volume is 432 cu. in. If the area of the entire surface of a similar polyedron is 75 sq. in., find its volume. *Ans.* 250 cu. in.

GENERAL EXERCISES

1. If from any point within a regular tetraedron perpendiculars are drawn to its faces, their sum equals an altitude of the tetraedron. (Compare this with Ex. 2, p. 96.)

2. A wedge whose altitude is 10 in. and edge 4 in., has a base that is a square having a perimeter of 36 in. Find the volume of the wedge.

3. A concrete pier for a railway bridge has dimensions as shown in the figure, the bases being rectangles with semicircles. Find the number of cubic yards of concrete in the pier. *Ans.* 37.8 − cu. yd.
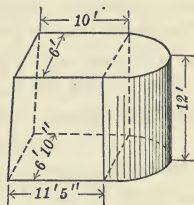
4. A railroad cut has the dimensions given in the figure, which shows the vertical section and three cross sections, one at each end and one in the middle. Find the number of cubic yards of earth removed in digging the cut.
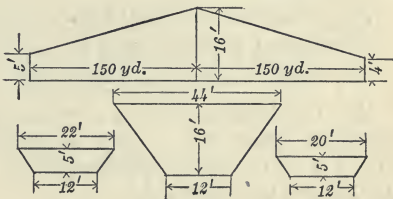
Ans. $7972\frac{2}{9}$ cu. yd.

SUGGESTION. The part on each side of the center is a prismatoid, and can be computed separately.

5. A tank of reinforced concrete is 160 ft. long, 100 ft. wide, and 10 ft. 6 in. deep, outside dimensions. The side walls are 8 in. thick at the top and 18 in. thick at the bottom, with the slope on the inside. The bottom is 6 in. thick. Find the number of cubic yards of cement used in building the tank, and the capacity of the tank in gallons.

6. The vertices of one regular tetraedron are the centers of the faces of another regular tetraedron. Find the ratio of the volumes of the two tetraedrons.

7. The three planes passing through the lateral edges of a triangular pyramid, and bisecting the base edges, meet in a common straight line.

8. The six planes passing through the edges of a tetraedron, and bisecting the opposite edges, meet in a common point.

9. The point of intersection of the planes in the preceding exercise is the center of gravity of the tetraedron. Prove that the center of gravity of a tetraedron divides the line from a vertex to the center of gravity of the opposite face in parts that are in the ratio of 3 : 1.

The center of gravity of a face is at the intersection of the medians of that face.

10. The corresponding edges of three similar tetraedrons are 3 in., 4 in., and 5 in., respectively. Find the corresponding edge of a similar tetraedron equal in volume to their sum.

11. By means of the prismatoid formula show that the volume of a truncated quadrangular prism whose opposite faces are parallel, is equal to the area of a right section times one-fourth the sum of the lateral edges.

SUGGESTION. Consider two of the parallel faces, as B_1 and B_2, as bases. Then M is the midsection between these.

Let A be the area of a right section having a side in B_1 equal to n, and altitude h, the distance between B_1 and B_2.
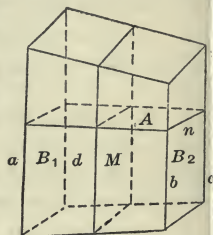
Then $B_1 = \frac{1}{2}(a+d)\,n$, $B_2 = \frac{1}{2}(b+c)\,n$, and $M = \frac{1}{2}\left\{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right\}n$

$\therefore\ V = \frac{1}{6}h\,[\frac{1}{2}(a+d)\,n+\frac{1}{2}(b+c)\,n+2\left\{\frac{1}{2}(a+b)+\frac{1}{2}(c+d)\right\}n]$

$\quad = \frac{1}{4}hn\,(a+b+c+d) = \frac{1}{4}A\,(a+b+c+d).$

12. The bottom of a bin of wheat is a rectangle 5 ft. by 12 ft. The top of the wheat is in a plane such that the depths at the corners are 6 ft., 5 ft., 3 ft., and 4 ft. respectively. Find the number of bushels in the bin if 1 bu. $= 1\frac{1}{4}$ cu. ft.

CHAPTER X

THE SPHERE AND POLYEDRAL ANGLES

GENERAL PROPERTIES

765. Sphere. A **sphere** is a closed surface all points of which are equally distant from a point within, called the **center** of the sphere.

A sphere is usually designated by a single letter at its center.

The **radius** of a sphere is the straight line connecting the center to a point in the surface. A straight line connecting two points in the surface and also passing through the center is a **diameter.**
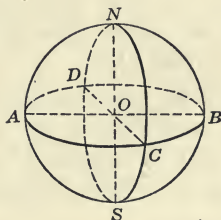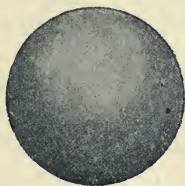
A sphere separates space into two parts so that any point that does not lie in the surface lies within the sphere or outside it.

The distance from the center to a point within a sphere is less than the length of the radius; while a point outside a sphere is at a greater distance from the center than the length of the radius.

766. Great Circle of a Sphere. It is evident from the definitions of a sphere and of a circle that a plane passing through the center of sphere intersects the sphere in a circle whose radius is equal to the radius of the sphere. Such a circle is called a **great circle** of the sphere.

767. The following facts follow readily from the definitions given:

(1) *Radii of the same sphere, or equal spheres, are equal.*

(2) *Diameters of the same sphere, or equal spheres, are equal.*

(3) *Spheres having equal radii, or equal diameters, are equal.*

(4) *All great circles of a sphere are equal.*

(5) *The points of intersection of any two great circles of a sphere are in a diameter.*

(6) *A great circle of a sphere bisects the sphere.*

(7) *A sphere may be generated by the revolution of a semicircle about its diameter.*

768. The Straight Line and the Sphere. In order to study the relative positions of a straight line and a sphere, pass a plane through the line and the center of the sphere. The line will then have the same relations to the sphere that it bears to the great circle in this plane.
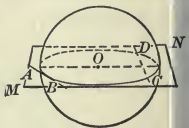
(1) *If the line intersects the great circle it cuts the circle in two points, and likewise the sphere.*

(2) *If the line is tangent to the great circle it has only one point in common with the sphere, and is perpendicular to the radius at this point.*

769. Definition. If a line has but one point in common with the sphere it is said to be **tangent to the sphere.**

<div align="center">EXERCISES</div>

1. A line perpendicular to a radius at its extremity is tangent to the sphere.

2. The locus of all lines tangent to a sphere at a point is a plane perpendicular to a radius at its outer extremity.

3. Find the locus of the centers of all spheres tangent to a given line at a given point.

4. Find the locus of the centers of all spheres of a given radius tangent to a fixed line.

5. The plane perpendicular to a tangent at its point of contact passes through the center of the sphere.

6. What is the locus of the vertices of the right angles of the right triangles having a given hypotenuse?
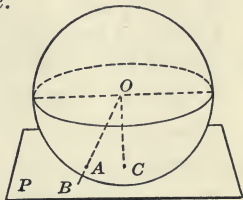
770. The Plane and the Sphere. If a plane has but one point in common with a sphere, it is said to be **tangent to the sphere.**

771. Theorem. *A plane perpendicular to a radius of a sphere at its outer extremity is tangent to the sphere.*

Given plane P perpendicular to the radius OC at its outer extremity C.

To prove plane P is tangent to the sphere O.

Give a proof similar to that of § 284.

772. Theorem. *A plane tangent to a sphere is perpendicular to the radius drawn to the point of contact.*

For if the plane was not perpendicular to the radius it would have a point less than the length of the radius from the center of the sphere.

773. Theorem. *If a plane is tangent to a sphere, the radius drawn to the point of tangency is perpendicular to the plane.*

EXERCISES

1. Two lines perpendicular to a radius at its outer extremity determine a plane tangent to a sphere.

2. Planes tangent to a sphere at the extremities of a diameter are parallel.

3. Explain how to construct a line tangent to a sphere at a point on the sphere. Explain how to construct a tangent plane at this point.

**4.** Explain how to construct a line and a plane tangent to a sphere and through an external point.

5. A point B is 22 in. from the center O of a sphere having a radius of 12 in. Find the distance from B to C, the point of tangency of a plane through B.

6. Find the locus of the centers of all spheres tangent to two given intersecting planes.

CIRCLES OF SPHERES

774. Theorem. *The section of a sphere made by a plane is a circle, whose center is the foot of the perpendicular from the center of the sphere to the plane.*

Given the sphere O cut by the plane P in the section ABN, also OQ perpendicular to plane P.
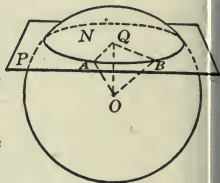
To prove that section ABN is a circle with center Q.

Proof. Draw QA and QB to any two points A and B on the section ABN. Also draw OA and OB.

$\triangle AQO$ and $\triangle BQO$ are congruent right triangles.   Why?

Then $AQ = BQ$. That is, any two points on section ABN are equidistant from Q.   Why?

∴ section ABN is a circle with center Q.   Why?

EXERCISES

1. If a plane that intersects a sphere is gradually moved from the center, describe the change in the intersection of the plane with the sphere.

2. A line that is tangent to a sphere is tangent to each section of the sphere formed by a plane containing the line.

3. The radius of a sphere is 14 in. Find the area of a section made by a plane 10 in. from the center.

4. The radius of a sphere is 12 in. If the area of a section made by a plane is 314.16 sq. in., find the distance from the center of the sphere to the plane.

5. How far from the center of a sphere, having a radius of 12 in., must a plane pass so that the section made by this plane shall be one-half the area of a great circle of the sphere?

6. In the same sphere or equal spheres, two sections equally distant from the center, are equal, and conversely. Compare with § 281.

7. What is the locus of the projections of a given point upon the planes passing through another point?

8. What is the locus of a point from which three planes can be drawn tangent to a given sphere and form a triedral angle all of whose diedral angles are right?

775. Definitions. A **small circle of a sphere** is the inter-
section of the sphere with any plane not
passing through its center.

The **axis of a circle** of a sphere is the
diameter of the sphere that is perpendicular
to the plane of the circle.

The **poles of a circle** of a sphere are the
extremities of the axis of the circle.
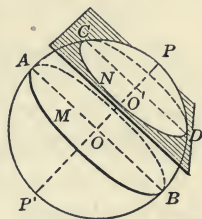
In the figure, AMB is a great circle, CND is a small circle, PP' is the
axis of each of these circles, and P and P' are the poles.

776. Theorem. *Through any three points on a sphere one and
only one circle of the sphere can be drawn.*

For the three points determine a plane.

777. Theorem. *Through any two points on a sphere a great
circle can be drawn.*

What third point is in the plane of the great circle?

How are the points situated when only one great circle can be drawn?
How when more than one can be drawn?

778. Theorem. *Parallel circles of a sphere have the same axis
and the same poles.*

779. Theorem. *If one great circle of a sphere
is perpendicular to another, either circle contains
the axis and poles of the other; and conversely.*
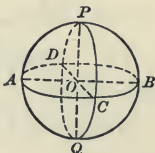
Use §§ 767 (5), 594.

EXERCISES

1. If the surface of the earth is considered a sphere, what kind of
circles are the equator, the meridians, and the parallels of latitude? What
is the axis of the equator? Of the parallels of latitude? What are the
poles of each of these?

2. Find the circumference of the parallel of latitude 45° north of the
equator. Of the parallel of latitude 60° north. (Use 4000 miles as the
radius of the earth.)

DISTANCE ON A SPHERE

780. By the **distance between two points** on a sphere is meant the length of the minor arc (§ 264) of the great circle through the two points. That this is the shortest path on the sphere between the two points can be proved.

If the two points are at the extremities of a diameter, the distance between them is a semicircumference of the great circle.

It should be noted that an arc of a great circle of a sphere here takes the place of a straight line in a plane in determining the distance. That is, the distance between two points in a plane is measured along the straight line joining them, while on a sphere the distance is measured along a great circle.

781. A **quadrant** of a sphere is one quarter of the circumference of a great circle of the sphere.

782. Polar Distance. The **polar distance** of a small circle of a sphere is the distance on the sphere from the nearer pole to a point in the circle.

The polar distance of a great circle may be taken from either of its poles to a point in the circle.

783. Theorem. *The polar distances of a circle of a sphere are equal.*

Given ⊙O' on sphere O with poles N and S.

To prove that all points in ⊙O' are equally distant from N, or S.

Proof. Take A and B any two points in ⊙O', and draw great circles through NAS and NBS; also draw lines AO', BO', AN, and BN.

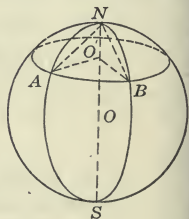Prove $\triangle AO'N \cong \triangle BO'N$, and therefore $AN = BN$.

Then $\qquad\qquad\qquad \widehat{AN} = \widehat{BN}.$ Why?

In all points in ⊙O' are equally distant from N.

∴ like manner prove for pole S.

784. Theorem. *On the same sphere or on equal spheres, the polar distances of equal circles are equal.*

785. Theorem. *The polar distance of a great circle is a quadrant; and conversely.*

786. Theorem. *The straight lines joining the points in the circle of a sphere to a pole are equal.*

EXERCISES

1. What is the name of the circle on the surface of the earth at a quadrant's distance from the north pole? At $23\frac{1}{2}°$ from the north pole? At $23\frac{1}{2}°$ from the south pole? At $23\frac{1}{2}°$ from the equator?

2. Show how to construct a circle on a sphere and at a given distance from a pole. Discuss for a small circle and for a great circle.

3. Show how to construct a circle of given radius on the surface of a sphere.

4. If the radius of a sphere is 10 in., find the radius of a circle having a polar distance of 45°. Of 30°.

787. Theorem. *If a point on a sphere is at a quadrant's distance from each of two other points, not the extremities of a diameter, on the sphere, it is a pole of the great circle through these two points.*

Given a point N on a sphere O, at a quadrant's distance from each of points A and B, and ABC the great circle through A and B.

To prove that N is the pole of $\odot ABC$.
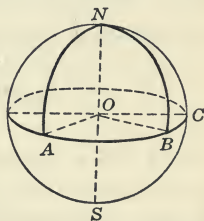
Proof. $\angle AON$ and $\angle BON$ are rt. $\measuredangle$. Why?

Then ON is perpendicular to plane of $\odot ABC$. § 571

And $\qquad SN$ is the axis of $\odot ABC$. Why?

$\qquad \therefore N$ is a pole of $\odot ABC$. Why?

EXERCISES

1. Show how to find the pole of a great circle on a sphere.

2. Show how to construct a great circle through two points on a sphere.

788. Circumscribed Polyedrons. A polyedron is said to be **circumscribed about a sphere** when the sphere is tangent to each face of the polyedron. The sphere is then said to be **inscribed in the polyedron.**

789. Inscribed Polyedrons. A polyedron is said to be **inscribed in a sphere** when all its vertices lie in the sphere. The sphere is then said to be **circumscribed about the polyedron.**
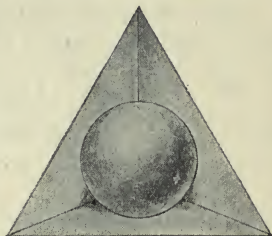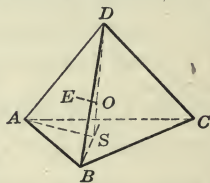
790. Theorem. *One, and only one, sphere can be inscribed in a given tetraedron.*

Given the tetraedron $ABCD$.

To prove that one sphere and only one can be inscribed in $ABCD$.

Outline of proof. Construct the planes bisecting the diedral angles AD and BD. These planes will intersect in a line DS.   Why?

Then DS is the locus of all points equally distant from faces ADC, ABD, and BDC.   Why?

Construct the plane bisecting the diedral angle AB.

This plane will intersect DS in some point O.   Why?

Then O is equally distant from the four faces of the tetraedron.   Why?

If a sphere is constructed with O as center and the perpendicular from O to any face, as OE, for radius, this sphere will be tangent to each face of the tetraedron, and therefore is an inscribed sphere.   § 789

To show that only one inscribed sphere can be constructed show that there is only one point O.

Compare this theorem and proof with the corresponding problem in plane geometry. (§ 325).

EXERCISE. Find the locus of the centers of all spheres tangent to the faces of a triedral angle.

791. Theorem. *One, and only one, sphere can be circum-scribed about a given tetraedron.*

Given the tetraedron $ABCD$.

To prove that one sphere and only one can be circumscribed about $ABCD$.

Outline of proof. Circumscribe a circle about $\triangle ABC$, and erect a perpendicular $O'Q$ to its plane at its center O'.
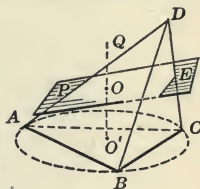
Then $O'Q$ is the locus of all points equally distant from A, B, and C. Why?

At the midpoint E of the edge CD construct a plane P perpendicular to CD.

Then plane P is the locus of all points equally distant from C and D. Why?

But plane P intersects $O'Q$ in a point O. Why?

Then O is equally distant from the four vertices of the tetraedron, and a sphere having O as center and OA as radius will pass through the four vertices A, B, C, and D. Why?

Therefore a sphere can be circumscribed about the tetraedron.

To show that only one sphere can be circumscribed about $ABCD$, show that there is only point O.

Compare this theorem and proof with the corresponding problem in plane geometry. (§§ 316, 317).

EXERCISES

1. What is the locus of points equidistant from four points not all in the same plane?

2. How many points are necessary to determine a sphere? How many points on a sphere are necessary to determine it if its center is given?

3. Prove that the planes that are the perpendicular bisectors of the edges of a tetraedron pass through a common point.

4. Prove that, if a circle is circumscribed about each face of a tetraedron, the perpendiculars to the respective faces at the centers of the circumscribing circles are concurrent.

5. One, and only one, sphere can be inscribed in or circumscribed about a cube.

792. Problem. *To find the radius of a given material sphere.*
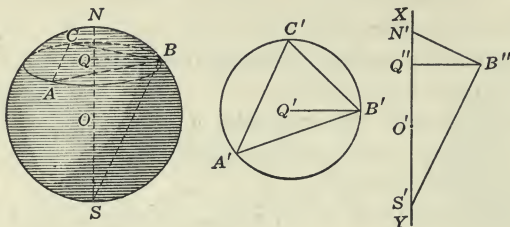
Given a material sphere O.

To find its radius.

Construction. Choose any point N on the sphere as a pole, and describe a convenient circle ABC.

Choose any three points on this circle as A, B, and C.

Construct in a plane $\triangle A'B'C'$ congruent to $\triangle ABC$.   § 25

Circumscribe a circle with center Q' about $\triangle A'B'C'$.

Draw $Q''B''$ equal to the radius $Q'B'$, and then construct $XY \perp Q''B''$ and through Q''.

With the compasses lay off $B''N'$ equal to BN, to meet XY at N', and construct $B''S' \perp B''N'$ at B'' and extend it to meet XY at S'.

Determine the bisecting point O' of $S'N'$.

Then $O'N'$ is the radius of the sphere O.

The proof is left to the student.

<div align="center">EXERCISES</div>

1. If $NB = 13$ in. and $QB = 12$ in., compute ON.   *Ans.*   16.9 in.

2. By means of an instrument called a spherometer, the distances QN and QB can be measured. Show how the radius ON can then be computed. Such an instrument is used practically in determining the radius of curvature of a spherical lens.

3. Find the radius of curvature of a spherical lens, that is, the radius of the sphere of which the surface of the lens is a part, if $QN = 0.125$ cm. and $QB = 2$ cm.   If $QN = 0.643$ cm. and $QB = 2.5$ cm.,
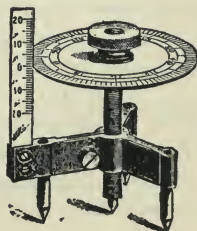
793. Sphere Inscribed in Cylinder or Cone. A sphere is said to be **inscribed in a cylinder,** or the cylinder **circumscribed about the sphere,** when the bases of the cylinder and all its elements are tangent to the sphere.

A sphere is said to be **inscribed in a cone,** or the cone **circumscribed about the sphere,** when the base of the cone and all its elements are tangent to the sphere.

794. Cylinder or Cone Inscribed in a Sphere. A cylinder is said to be **inscribed in a sphere,** or the sphere **circumscribed about the cylinder,** when its bases are circles of the sphere.

A cone is said to be **inscribed in a sphere,** or the sphere **circumscribed about the cone,** when its base is a circle of the sphere and its vertex lies on the sphere.

795. Theorem. *All the tangents to a sphere from a given point are equal, and their points of contact are in a circle of the sphere.*

Given sphere O, and point P outside the sphere.

To prove that the tangents to the sphere from the point P are equal, and that their points of contact lie in a circle.

Outline of proof. Let PB be one tangent. Then PB is tangent to a great circle determined by P, B, and O.   Why?

Therefore $\angle PBO$ is a rt.$\angle$.   § 288

Pass a plane through $B \perp PO$. This will intersect the sphere in a circle with center O'.

Further, a line drawn from P to any point, as C, in the $\odot O'$ is tangent to the sphere O, for $\triangle OCP \cong \triangle OBP$, and therefore $PC \perp OC$.   Why?

Show that no tangent from P could have its point of contact outside of $\odot O'$.

That the tangents are all equal follows from § 584.
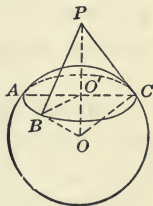
RELATIVE POSITIONS OF SPHERES

**796.** The relative positions of two spheres are analogous to the relative positions of two circles. (See §§ 291-296.)

797. Line of Centers. The line joining the centers of two spheres is the line of centers.

798. Tangent Spheres. If two spheres are tangent to the same plane at the same point, they are **tangent spheres.** They may be tangent internally or externally.

Spheres A and B are tangent internally; and A and C externally.
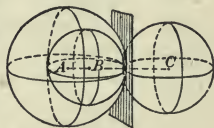
799. Concentric Spheres. Spheres that have a common center are **concentric spheres.**

800. Theorem. *The intersection of two spheres is a circle, whose center is in a straight line joining the centers of the spheres and whose plane is perpendicular to that line.*

Given two intersecting spheres O and O'.

To prove that the intersection of the spheres is a circle whose center is in OO' and whose plane is perpendicular to OO'.
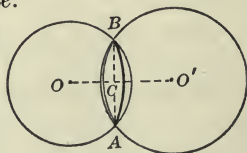
Outline of proof. Through O, O', and any point A of the intersection, pass a plane. This plane intersects the two spheres in two great circles intersecting in two points. Why?

If A and B are these points, then OO' is the perpendicular bisector of AB.   § 295

Show that if the entire figure is revolved about OO', the great circles generate the spheres, and point A generates a circle with center C and radius CA, which is common to the spheres.

Also show that the plane of the circle generated by A is perpendicular to OO'.

Further, show that if any point of the intersection was outside of this circle, then the spheres would coincide. (§ 791).

801. Theorem. *If two spheres are tangent to each other, the line of centers passes through the point of contact.*

Proof similar to that of § 296.

1. State and prove exercises concerning the sphere analogous to Exercises 1 to 5, page 115.

2. What is the locus of points at a distance r from a given point O, and at a distance r' from a given point O'?

3. Find the center of a sphere which contains a given circle and also contains a given point not lying in the plane of the circle.

4. Two spheres with radii 34 and 50, respectively, intersect. If the distance between their centers is 56, find the radius of their circle of intersection, and the distance from the center of each sphere to the center of the circle of intersection.

5. The line of contact of a sphere inscribed in a circular cone with the conical surface of the cone, is a small circle of the sphere.

6. The line of contact of a sphere inscribed in a circular cylinder with the cylindrical surface, is a great circle of the sphere.

MEASUREMENT OF THE SPHERE, AREA

802. Area of a Sphere. By the **area of a sphere** is meant the numerical measure of the surface, that is, the number of units of area in it.

As with the cylinder and cone, a sphere is a curved surface, and a plane unit of area cannot be applied directly to it to find its numerical measure. Its area can be determined as a limit. It is based upon (7) of § 767, which states that a sphere is generated by revolving a semicircle about its diameter, and the following statement which is accepted without proof:

If in a semicircle generating a sphere, one-half of a regular polygon be inscribed so that a vertex lies at each end of the diameter, then, as the number of the sides of the inscribed semipolygon is indefinitely doubled, the area generated by the semipolygon approaches the area of the sphere as a limit.
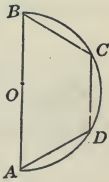
803. Theorem. *The area of the surface generated by a segment of a straight line revolving about an axis, in its plane but not perpendicular to it and not intersecting it, is equal to the projection of the segment upon the axis multiplied by the circumference of the circle whose radius is the perpendicular erected at the midpoint of the segment and terminated by the axis.*
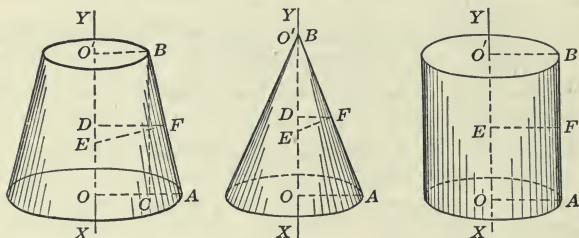
Given the segment AB revolving about the axis XY in its plane.

To prove that $S = OO' \times 2\pi \times EF$, where S denotes the area of the surface generated, OO' is the projection of AB upon XY, and FE is the perpendicular at the midpoint of AB.

Proof. Three cases arise:

CASE I, in which AB is oblique to and does not meet XY. The surface generated is the lateral surface of a frustum of a right circular cone.

The proof of this is given in § 729.

CASE II, in which AB is oblique to and meets XY. The surface generated is the lateral surface of a right circular cone.

$$S = \pi \times OA \times AB. \qquad \text{§ 719}$$

Show that $\quad \triangle DEF \sim \triangle OAB.$

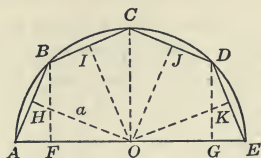Then $\quad AB : OB = EF : DF$, and $AB \times DF = OB \times EF.$

Since $OB = OO'$ and $OA = 2DF$, $AB \times OA = 2OO' \times EF.$

$$\therefore S = OO' \times 2\pi \times EF. \qquad \text{§ 111}$$

CASE III, in which AB is parallel to XY. The area generated is the lateral surface of a right circular cylinder.

Give proof.

804. Theorem. *The area of a sphere is equal to four times the area of a great circle of the sphere.* $S = 4\pi r^2$.

Given the area S of a sphere generated by the revolution of the semicircle O of radius r, about the diameter AE.

To prove that $S = 4\pi r^2$.

Proof. Inscribe in the semicircle one-half of a regular polygon of an even number of sides, as $ABCDE$. Let A denote the apothem of the polygon, and let F, O, and G be the projections of B, C, and D respectively upon the diameter AE.

For each side, the apothem is the perpendicular bisector. Why?

Then   area generated by $AB = AF \times 2\pi a$.   § 803

Similarly   area generated by $BC = FO \times 2\pi a$.

<div align="center">Etc.</div>

Adding these equations and denoting by S' the area generated by the revolution of the semipolygon $ABCDE$,

$$S' = (AF + FO + \cdots)2\pi a.$$

But $(AF + FO + \cdots) = AE = 2r$.

Then $S' = 4\pi r a$, which is always true as the number of the sides of the polygon is continuously doubled.

Further $S' \rightarrow S$ and since $a \rightarrow r$, $4\pi r a \rightarrow 4\pi r^2$. §§ 802, 490, 485 (2)

$$\therefore \ S = 4\pi r^2. \qquad \text{§ 485 (1)}$$

Show that other forms of the formula for the area of a sphere are $S = 2\pi r d$, and $S = \pi d^2$.

805. Theorem. *The areas of two spheres are to each other as the squares of their radii; or, as the squares of their diameters.*

For, if S, and S' are the areas of two spheres,

$$\frac{S}{S'} = \frac{4\pi r^2}{4\pi r'^2} = \frac{r^2}{r'^2}, \text{ and } \frac{S}{S'} = \frac{\pi d^2}{\pi d'^2} = \frac{d^2}{d'^2}.$$

EXERCISES

1. Given the area of a sphere, to find its radius and its diameter.

2. Find the area of a sphere whose radius is 5. Of a sphere whose diameter is 16. Of a sphere whose circumference is 24 in.

3. Find the radius of a sphere whose area is 100 sq. in.

4. How many square feet of tin will it take to roof a hemispherical dome 40 ft. in diameter?

5. Find the diameter of a sphere that has the same area as a cube that is 8 in. on an edge.

6. Find the area of a sphere that is circumscribed about a cube that is $12\sqrt{3}$ in. on an edge.   *Ans.* 4071.5 sq. in.

7. The number of square inches in the area of a sphere is equal to the number of linear inches in the circumference of a great circle of the sphere. Find the radius of the sphere.

8. Show that, if a cylinder is circumscribed about a sphere, the lateral area of the cylinder is equal to the area of the sphere. Also show that the area of the sphere equals two-thirds the total area of cylinder.

9. The radius of the earth is 3960 miles and the radius of the moon is 1080 miles. Find the ratio of their areas.

10. Show how to determine the distance apart the points of a compass must be placed to draw a great circle upon a material sphere that has a diameter of 14 in. So as to draw a circle having a radius of 5 in.

11. What is the locus of the centers of all spheres tangent to a given plane at a given point?

12. What is the locus of the centers of all spheres passing through three given points?

13. What is the locus of points at a distance r_1 from a point O_1 and at a distance r_2 from another point O_2? Discuss.

14. Find the area of a sphere circumscribed about a tetraedron whose edge is 4 in.

15. The radii of two intersecting spheres are 10 in. and 12 in., respectively, and the distance between their centers is 16 in. Find the radius of the circle of intersection of the two spheres.

16. Three equal spheres each tangent to the other two rest on a plane. A fourth sphere rests on the first three spheres. Find the distance from the center of the fourth sphere to the plane, if each sphere has a radius of 8 in.   *Ans.* 21.064 in.
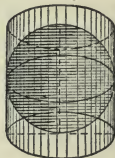
806. Zones. The portion of the surface of a sphere included between two parallel planes is called a **zone.**

It is evident that a zone can be considered as generated by an arc of a great circle.

The circles made by the two parallel planes are called the **bases** of the zone, and the distance between the planes is the **altitude** of the zone.

If one of the planes is tangent to the sphere the zone is called a **zone of one base.**

The zones on the surface of the earth are included between certain parallels of latitude. The torrid and temperate zones are zones of two bases, and the frigid zones are zones of one base.

807. Theorem. *The area of a zone of a sphere is equal to the product of the altitude of the zone and the circumference of a great circle of the sphere; that is, area* $= 2\pi rh$, *where r denotes the radius of a great circle and h the altitude of the zone.*

The proof is similar to that of § 804.

808. Theorem. *The areas of zones of the same sphere or of equal spheres are to each other as their altitudes.*
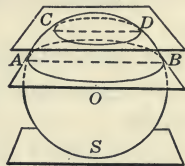
EXERCISES

1. Find the area of a zone of one base and having an altitude of 5 ft. on a sphere 8 ft. in diameter.

2. Find the radius of a sphere, on which a zone of altitude 5 in. has an area of 100 sq. in.

3. The diameter of a sphere is 12 in., and is divided into four equal parts by parallel planes. Find the area of each zone.

4. Find what part of the earth's surface is north of the parallel 45° north. North of the parallel 60° north.

5. Show that one-half of the earth's surface lies within 30° of the equator.

6. What part of the earth's surface could be seen from a point whose distance from the surface is equal to the radius? Twice the radius?

809. Lunes. That portion of a sphere between the halves of two great circles of the sphere is called a **lune**.

The diedral angle between the planes of the great circles bounding the lune is the **angle of the lune**.
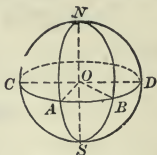
From § 589 it follows that the angle of a lune is measured by the plane angle of the diedral angle formed by the planes of the great circles bounding the lune.

810. Comparison of Lunes. If, on the same sphere, one lune is placed on another with one bounding semicircle in common, the other bounding semicircles either coincide, or one lies wholly within the lune bounded by the other. It is evident then that:

(1) *On the same sphere or equal spheres, two lunes having equal angles are equal; and conversely.*

(2) *On the same sphere or equal spheres, two lunes are in the same ratio as their angles.*

811. Theorem. *The area of a lune is given by the formula* $L = \dfrac{u}{360}(4\pi r^2)$, *where L denotes the area of the lune, u the angle of the lune in degrees, and r the radius of the sphere.*

To prove this, consider the sphere as a lune whose angle is 360°, then by § 810 (2), $\dfrac{L}{4\pi r^2} = \dfrac{u}{360}$, or $L = \dfrac{u}{360}(4\pi r^2)$.

EXERCISES

1. Find the area of a lune whose angle is 30° on a sphere 8 ft. in diameter.

2. Find the part of the earth's surface between the equator and the parallel 30° north, and between the meridians 120° west and 150° west.

3. The chord of the polar distance of a circle is 8 in. If the radius of the sphere is 10 in., what is the radius of the circle? What is the area of the zone cut off by the circle?

4. Find the altitude of a zone whose area equals the area of a great circle of the sphere.

5. A plane passes through a sphere bisecting at right angles a radius of the sphere. Compare the areas of the two parts of the sphere.

6. If h is the height of P above the surface of the earth and r the radius, then S, the extent of the surface of the earth visible from P, is given by
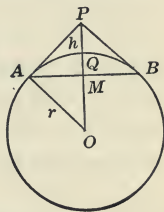
the formula $S = \dfrac{2\pi r^2 h}{r+h}$.

$S = 2\pi r \times MQ$.   § 807

But $MQ = r - OM$, and $OM : OA = OA : OP$.

Then $OM = \dfrac{r^2}{r+h}$, and $MQ = r - \dfrac{r^2}{r+h} = \dfrac{rh}{r+h}$.

$\therefore S = 2\pi r \times \dfrac{rh}{r+h} = \dfrac{2\pi r^2 h}{r+h}$.

7. How many miles above the surface of the earth is a point from which one-fourth of the surface can be seen?   *Ans. r.*

8. From a point 6 ft. from a sphere one-fourth of its area is visible. Find the radius of the sphere.

9. An ash tray is in the form of the zone of a sphere. If the tray is 1 in. deep and $3\frac{1}{2}$ in. in diameter at the top, find the diameter of the blank from which it is pressed.

10. A zone of one base has the same area as a circle whose radius is the chord of the generating arc of the zone, that is, the area of the zone generated by arc BC revolving about DC is equal in area to a circle having chord BC for a radius.
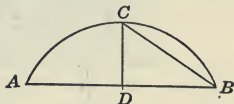
11. Show that the area of a zone of one base is given by the formula

$$S = \tfrac{1}{4}\pi (w^2 + 4h^2) = \pi [(\tfrac{1}{2}w)^2 + h^2],$$

where S denotes the area, w the diameter of the base of the zone and h the altitude.

12. In a practical handbook, the following rule is given as *nearly* correct. Show that it is accurate. The area of a flanged spherical segment is equal to a circle of radius equal in length to the line drawn from the top of the segment to the edge of the flange, that is equal to a circle of radius AB.

SUGGESTION.   Area of zone $= \pi [(\tfrac{1}{2}w)^2 + h^2]$.

Area of flange $= \pi [(\tfrac{1}{2}d)^2 - (\tfrac{1}{2}w)^2]$.

Area of zone and flange $= \pi [(\tfrac{1}{2}d)^2 + h^2] = \pi \overline{AB}^2$.
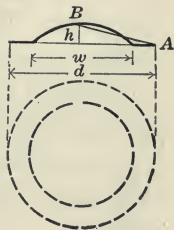
VOLUME OF A SPHERE

**812.** By the **volume of a sphere** is meant the volume within the sphere.

813. If a regular polyedron, for instance a cube, is circumscribed about a sphere, and lines drawn from each vertex to the center of the sphere, these lines may be taken as the lateral edges of pyramids having their vertices at the center of the sphere and having the faces of the polyedron as bases. The volume of the polyedron can then be expressed as the volume of these pyramids.
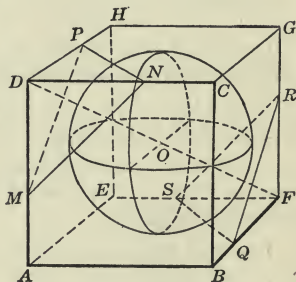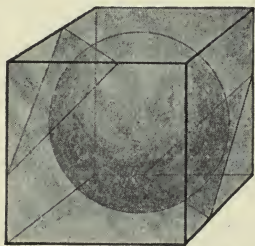
Further, if tangent planes be drawn where each lateral edge intersects the sphere, a circumscribed polyedron of a greater number of faces will be formed. It is evident that this process can be continued indefinitely, thus forming polyedrons whose areas and volumes steadily decrease, and can be made to differ as little as desired from the area and volume, respectively, of the sphere. The following statement can then be accepted.

814. *If each vertex of a circumscribed polyedron is joined to the center of the sphere, and tangent planes are drawn at the points where these lines intersect the sphere, then, as the number of faces is increased indefinitely, the area and the volume of the polyedron approach as limits the area and the volume, respectively, of the sphere.*

815. Theorem. *The volume of a sphere is equal to the product of its area by one-third of its radius.*
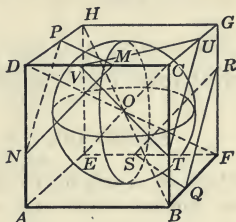
Given a sphere O of radius r and area S.

To prove that the volume $V = S \times \frac{1}{3}r$.

Proof. Circumscribe about the sphere any polyedron, for instance a cube, and denote the surface of the polyedron by S'.

Since the polyedron may be considered as composed of pyramids having the faces of the polyedron as bases and the center of the sphere as common vertex, the volume V' of the polyedron is
$$V' = S' \times \frac{1}{3}r.$$

If the number of faces of the polyedron is indefinitely increased as described in § 813,
$$V' \to V, \ S' \to S, \text{ and } S' \times \tfrac{1}{3}r \to S \times \tfrac{1}{3}r. \quad \text{§§ 814, 485 (2)}$$

But $V' = S' \times \frac{1}{3}r$ is always true as the number of faces is increased.
$$\therefore V = S \times \tfrac{1}{3}r. \qquad \text{§ 485}$$

816. Theorem. *The volume V of a sphere of radius r and diameter d is given by the formulas:* $V = \frac{4}{3}\pi r^3$ *and* $V = \frac{1}{6}\pi d^3$.

For $V = \frac{1}{3}rS$ by § 815. But $S = 4\pi r^2$ or πd^2 by § 804.
$$\therefore V = \tfrac{4}{3}\pi r^3, \text{ and } V = \tfrac{1}{6}\pi d^3.$$

817. Theorem. *The volumes of two spheres are to each other as the cubes of their radii, or as the cubes of their diameters.*

Let V_1, r_1, d_1 and V_2, r_2, d_2 be the volumes, radii and, diameters, respectively.

Then
$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi r_1^3}{\frac{4}{3}\pi r_2^3} = \frac{r_1^3}{r_2^3}, \text{ or } \frac{V_1}{V_2} = \frac{\frac{1}{6}\pi d_1^3}{\frac{1}{6}\pi d_2^3} = \frac{d_1^3}{d_2^3}.$$

818. Theorem. *The prismoid formula holds for a sphere.*

In the formula, $V = \frac{1}{6}h(B_1 + B_2 + 4M)$ of § 751, $h = 2r$, $B_1 = 0$, $B_2 = 0$, $4M = 4\pi r^2$. $\therefore V = \frac{4}{3}\pi r^3$.

819. Theorem. *The volume of a sphere is equal to two-thirds the volume of the circumscribed right cylinder.*

EXERCISES

1. Consider the surface of a sphere divided as shown in the figure. Can you conclude that the volume is as given in § 815?

2. Show that the volumes of the right circular cylinder, the sphere, and the right circular cone with dimensions as given in the figure, are in the ratio of $3 : 2 : 1$.

NOTE. Archimedes (287-212 B.C.) who was the greatest mathematician of antiquity, proved the theorem of Exercise 2. This theorem may well be regarded as one of the most beautiful in elementary mathematics. Archimedes proved many of the most important theorems in regard to the sphere.

3. State a formula for finding the radius of a sphere when the volume is given. One for finding the diameter when the volume is given.

4. A ball of diameter 2 in. is inside a right circular cylinder of diameter and altitude each 2 in. How many cubic inches of water can be poured into the cylinder?

5. Find the weight of a spherical cast iron ball of diameter 6 in., if cast iron weighs 0.26 lb. per cubic inch.

6. Find the volume of a hollow spherical shell, the outer diameter being 8 in. and the thickness 2 in.   *Ans.*   234.6 cu. in.

7. How many spherical bullets $\frac{2}{5}$ of an inch in diameter can be made from 5 pounds of lead, if lead weighs 0.412 pounds per cubic inch?

8. The volume of two spheres are in the ratio of $512 : 27$. Find the radius of each if the sum of their radii is 44 in.

9. A water tank, having a total length of 6 ft. and a diameter of 18 in., is in the form of a right circular cylinder with two hemispherical ends. Find its capacity in gallons.   *Ans.*   72.7+ gal.
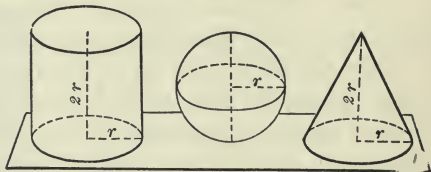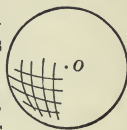
820. Spherical Sector. If a sphere is generated by the revolution of a semicircle, the solid generated by a sector whose arc is a part of this semicircle is called a **spherical sector.**

A spherical sector generated by a circular sector revolving about one of its bounding radii is called a **spherical cone.**
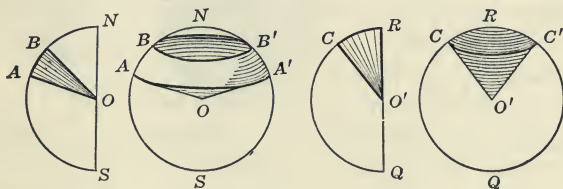
The zone generated by the arc of the generating sector is called the **base** of the spherical sector.

821. Theorem. *The volume of a spherical sector is equal to one-third the product of the area of its base, and its radius.*

EXERCISES

1. A right circular cone having a vertex angle of 60° has its vertex at the center of a sphere 4 ft. in diameter. What is the volume cut from the sphere by the cone?

2. Find the volume of a spherical sector cut from a sphere 4 ft. in diameter, and whose base is a zone having an altitude of 8 in.

822. Spherical Segment. The portion of the volume of a sphere contained between two parallel planes that intersect the sphere or are tangent to it, is called a spherical segment.

The sections of the sphere made by the two parallel planes are the **bases** of the spherical segment.

If one of the parallel planes is tangent to the sphere, the segment is called a **spherical segment of one base.**

The distance between the parallel planes is the altitude of the spherical segment.
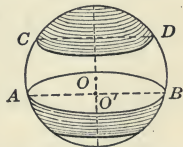
823. Theorem. *The volume of a spherical segment is given by the formula* $V = \frac{1}{2}\pi h(r_1^2 + r_2^2) + \frac{1}{6}\pi h^3$, *where h denotes the altitude, and r_1 and r_2 the radii of the bases of the segment.*

Given the spherical segment with altitude h, and bases ADB and EHF having radii r_1 and r_2 respectively.

To prove $V = \frac{1}{2}\pi h(r_1^2 + r_2^2) + \frac{1}{6}\pi h^3$.

Proof. Let $OG = x$,

V = volume of spherical sector formed by revolving OBF + volume of cone $O\text{-}ADB$ − volume of cone $O\text{-}EHF$. Why?

(1) $V = \frac{2}{3}\pi r^2 h + \frac{1}{3}\pi r_1^2(h+x) - \frac{1}{3}\pi r_2^2 x$. Why?

To reduce to the form desired it is necessary to eliminate r and x, and this requires two other equations. They are:

(2) $r^2 = r_1^2 + (h+x)^2$, and (3) $r^2 = r_2^2 + x^2$. Why?

From (2) and (3) $x = \dfrac{r_2^2 - r_1^2 - h^2}{2h}$.

Substituting this value of x in (3),

$$r^2 = \frac{h^4 + r_1^4 + r_2^4 + 2r_1^2 h^2 + 2r_2^2 h^2 - 2r_1^2 r_2^2}{4h^2}.$$
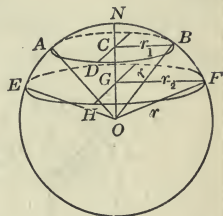
Substituting these values of x and r^2 in (1) and simplifying,
$$V = \frac{1}{2}\pi h(r_1^2 + r_2^2) + \frac{1}{6}\pi h^3.$$

Remark. The same formula is derived if the center of the sphere is within the spherical segment.

If the spherical segment has but one base, $r_1 = 0$, and the formula becomes $V = \frac{1}{2}\pi h r_2^2 + \frac{1}{6}\pi h^3$.

The student should carry out the work in each of these cases.

### EXERCISES

1. What part of the volume of the earth is between the planes cutting its surface in the parallels of 30° and 45° north latitude? What part is north of a plane cutting the surface in the parallel of 60° north latitude?

2. In the formula for the volume of a spherical segment, solve for each letter so far as possible.

824. Spherical Wedge. That portion of the volume of a sphere contained between the planes of two great semicircles is called a **spherical wedge.** Its curved surface is its **base,** and is, evidently, a lune.
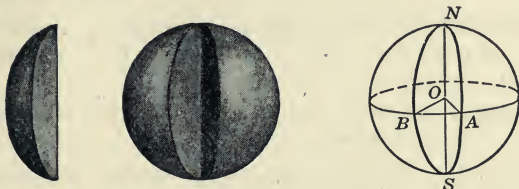
The solid *A-NS-B* is a wedge. The lune between the semicircles *NAS* and *NBS* is the base of the wedge. The angle *AOB* of the wedge is the angle of its lune.

825. Theorem. *The volume of a spherical wedge is equal to one-third the product of the area of its base and its radius.* That is, $V = \dfrac{u}{360}(\tfrac{3}{4}\pi r^3)$, *where r denotes the radius of the sphere and u the angle of the lune in degrees.*

By a discussion similar to that for the volume of a sphere, the volume of a wedge is given by the formula $V = \tfrac{1}{3}Lr$, where *L* is the area of the lune.

But $L = \dfrac{u}{360}(4\pi r^2)$ by § 811. ∴ $V = \dfrac{u}{360}(\tfrac{4}{3}\pi r^3)$.

EXERCISES

1. Find the volume of a spherical wedge whose angle is 20°, cut from a sphere whose radius is 12 in.

2. Find the angle of a spherical wedge if its volume is 418.88 cu. in., and the radius of the sphere is 10 in.

3. A circular flower bed is in the form of a spherical segment that is 25 ft. in diameter and $2\frac{1}{2}$ ft. in altitude. How many loads of dirt did it take to build it up if one load is $1\frac{1}{2}$ cu. yd.?

4. The figure is a vertical cross section of a casting, the inner and outer "skins" being spherical zones. Find the weight of metal at 0.35 lb. per cubic inch necessary to make the casting.   *Ans.* 176 lb.
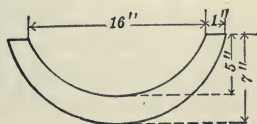
5. Find the volume of the segment between two parallel planes 6 in. apart that cut a sphere having a radius of 12 in., if one plane passes 2 in. from the center. There are two cases: (1) when the center of the sphere lies outside the segment, and (2) when the center lies in the segment. *Ans.* (1) 2186.6 cu. in.; (2) 2638.9 cu. in.

6. The number of square inches in the area of a certain sphere is equal to four times the number of cubic inches in its volume. Find the radius of the sphere.

7. A copper ball to be used as a float must contain 22,449.4 cu. in. What must be its diameter? *Ans.* 35 in.

8. Derive the formula for the volume of a sphere from the formula for the volume of a spherical sector.

9. Derive the formula for the volume of a sphere from the formula for the volume of a spherical segment.

SUGGESTION. Let $r_1 = 0$ and $r_2 = r$ and obtain the volume of a hemisphere.

10. If oranges $2\frac{1}{2}$ in. in diameter are selling for 30 cents a dozen, what should be the price for oranges 3 in. in diameter?

11. The water tank shown in the figure is in the form of a right circular cylinder with a hemisphere at the bottom. Find the capacity in gallons if the diameter is 22 ft. and the altitude of the cylinder is 24 ft. 6 in.

12. Find how many square feet of sheet steel in the tank if it is open at the top.

13. The roof over the tank is third pitch and projects 2 ft. at the eaves. Find the number of square feet in it.

14. A shop received an order for 100 twelve-inch steel balls to be used in crushing cement in a cement mill. They were made from $8\frac{1}{2}$-inch round steel shafting. What length of shaft was taken for each ball if 5% was allowed for scale?

15. A round or button-head machine screw has a head in the form of a spherical segment of one base. Show that the volume of the head is given by the formula: $V = \pi h(\frac{1}{8}w^2 + \frac{1}{6}h^2)$, where w is the diameter of the head and h its height.

16. Find the volume of the head of a round head machine screw if the diameter of the head is 0.731 in. and the height is 0.279 in.

Ans. 0.070 cu. in.

SPHERICAL ANGLES

826. Angles Formed by Arcs. The angle formed by two arcs of circles is the plane angle formed by the lines tangent to the arcs at the point of intersection.

Thus the angle formed by the arcs AB and CB is the angle formed by the tangent lines DB and EB.
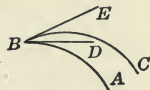
According to this definition, any two intersecting circles on a sphere form angles. In this text, however, only those angles formed by great circles are considered.

827. Spherical Angle. The angle between two great circle arcs is called a **spherical angle.** The point where the two arcs meet is the **vertex** of the angle, and the arcs are the **sides** of the angle.

The terms *right, acute,* and *obtuse* are applied to spherical angles and have the same meaning as with plane angles.

828. Since the planes of two great circles intersect in a diameter (§ 767), and the tangents at the points of intersection are perpendicular to this diameter (§ 288), the angle formed by the tangents is the plane angle of the diedral angle formed by the planes of the great circles § 588. It follows then from § 826 that:
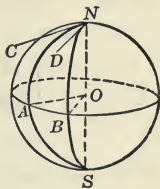
A spherical angle equals in degrees the diedral angle between the planes of its sides.

Spherical $\angle ANB = \angle CND = °$diedral $\angle A\text{-}NS\text{-}B$.

EXERCISES

1. Form definitions for the following when applied to spherical angles: adjacent, complementary, supplementary, vertical.

2. Prove that the sum of all the spherical angles about a point on a sphere is equal to 360°.

3. At what angle does a meridian of the earth intersect the equator?

4. Prove that vertical spherical angles are equal.

829. Theorem. *A sperical angle is equal in degrees to the arc of the great circle described from its vertex as a pole, and included between its sides, produced if necessary.*

Given spherical $\angle APB$, $\widehat{AB}$ an arc of a great circle whose pole is P, and which is included between the sides AP and BP.

To prove that spherical $\angle APB = °\widehat{AB}$.

Proof. Draw PO, AO, and BO.
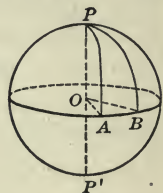
PO is perpendicular to plane of $\widehat{AB}$.   § 775

Then $OA \perp PO$ and $BO \perp PO$.   Why?

And $\angle AOB$ is the plane angle of diedral $\angle A\text{-}OP\text{-}B$.   Why?

Hence diedral $\angle A\text{-}OP\text{-}B = °\angle AOB$.   § 589 (3)

But $\angle AOB = °\widehat{AB}$.   § 305

$\therefore$ spherical $\angle APB = °\widehat{AB}$.   § 111
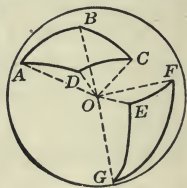
SPHERICAL POLYGONS AND POLYEDRAL ANGLES

830. Spherical Polygons. A portion of a sphere bounded by three or more arcs of great circles is called a **spherical polygon.** The bounding arcs are the **sides** of the polygon, the spherical angles formed by the sides are the **angles** of the polygon, and their vertices the **vertices** of the polygon.

831. A **diagonal** of a spherical polygon is the arc of a great circle joining any two vertices not adjacent.

It is evident that a side of a spherical polygon can be greater than a semicircle, but in this text only those will be considered whose sides are each less than a semicircle.

832. A **spherical triangle** is a spherical polygon of three sides.

The terms *right, acute, obtuse, isosceles,* and *equilateral* are applied to spherical triangles and have the same meaning as with plane triangles. It will be found, however, that a spherical triangle may have one, two, or three right or obtuse angles.

833. Central Polyedral Angles. It follows from the definition of a spherical polygon that the planes of the sides of the polygon pass through the center of the sphere. These planes form a polyedral angle, called the **corresponding central polyedral angle** of the spherical polygon. This polyedral angle and the spherical polygon are so related that it is convenient to consider them in connection with each other.
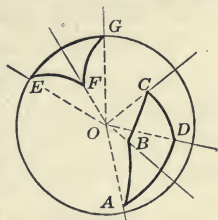
It follows directly from §§ 305 and 829 that:

(1) *The sides of a spherical polygon are equal in degrees to the face angles of the polyedral angle.*

(2) *The angles of a spherical polygon are equal in degrees to the diedral angles of the polyedral angle.*

Because of these relations it follows that to each property of a polyedral angle there is a corresponding property of the spherical polygon, and *vice versa*. One of these properties can be stated when the other is given by making the substitution indicated in the following:

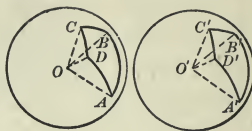| SPHERICAL POLYGON	POLYEDRAL ANGLE
Sides	Face angles
Angles of polygon	Diedral angles
Vertices of polygon	Edges
Center of sphere	Vertex

In this manner a theorem relating to spherical polygons can readily be worded so as to apply to polyedral angles, and *vice versa*.

834. A **convex spherical polygon** is a spherical polygon no side of which, when produced, will enter the polygon.

It is evident that the corresponding central polyedral angle of a convex spherical polygon is a convex polyedral angle. (See § 605.)

835. Two spherical polygons on the same sphere or equal spheres are **congruent** if their corresponding central polyedral angles are congruent. (See § 607.)

The spheres O and O' are equal and the polyedral angles having their vertices at the centers of the spheres are congruent. Then the corresponding spherical polygons $ABCD$ and $A'B'C'D'$ are congruent.

836. Symmetric Polyedral Angles and Spherical Polygons. Two polyedral angles are **symmetric** if the face angles and diedral angles of one are respectively equal to those of the other but arranged in reverse order.

The triedral angles $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are symmetric, for diedral angles AV, BV, and CV are equal respectively to diedral angles $A'V'$, $B'V'$, and $C'V'$; and the face angles AVB, BVC, and CVA are equal respectively to face angles $A'V'B'$, $B'V'C'$, and $C'V'A'$.
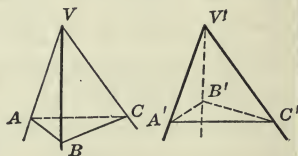
It is to be noticed that if the triedral angles are viewed from the vertices V and V', then the parts of the triedral angle $V\text{-}ABC$ are arranged in the reverse order to the parts of $V'\text{-}A'B'C'$.

If the two triedral angles were made of pieces of cloth sewed together, then, if one were turned "inside out," they could be made to coincide. They would then be congruent polyedral angles.

A pair of gloves are analogous to two symmetric polyedral angles. All parts of one are equal to corresponding parts of the other, but are arranged in reverse order. Right and left parts of animals are symmetric to each other in this sense. Numerous such illustrations of symmetry can be found in nature and in the arts.

837. Two spherical polygons on the same sphere or an equal spheres are **symmetric** if their corresponding central polyedral angles are symmetric.

In the figure, spherical triangles ABC and $A'B'C'$ are symmetric for their corresponding central triedral angles are symmetric.
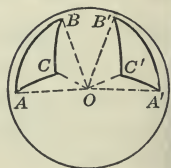
838. The two polyedral angles formed by a pyramidal surface are symmetric; and, if a sphere is drawn with its center at the vertex of the pyramidal surface, then the pyramidal surface will intersect the sphere in two symmetric spherical polygons.

It is evident that the parts of the two nappes of the pyramidal surface with vertex O are arranged in reverse order, then the corresponding spherical polygons $ABCD$ and $A'B'C'D'$ are symmetric.

Of course, symmetric spherical polygons can lie in any position on the sphere, for a figure on a sphere can be moved about without changing its shape.

839. Theorem. *In any spherical triangle, the sum of two sides is greater than the third side.*

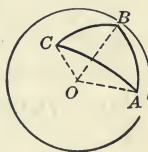**Given** spherical $\triangle ABC$ on sphere O.

To prove that $\widehat{AB} + \widehat{BC} > \widehat{AC}$.

Proof. $\angle AOB + \angle BOC > \angle AOC.$ § 608

But $\angle AOB = {}^{\circ}\widehat{AB}$, $\angle BOC = {}^{\circ}\widehat{BC}$, and $\angle AOC = {}^{\circ}\widehat{AC}$.   Why?

$$\therefore \widehat{AB} + \widehat{BC} > \widehat{AC}.$$ § 111

840. Theorem. *In any convex spherical polygon, the sum of the sides is less than a great circle of the sphere, or less than 360°.*

$$\angle AOB + \angle BOC + \angle COD + \angle DOA < 360°. \text{§ 609}$$
$$\therefore \widehat{AB} + \widehat{BC} + \widehat{CD} + \widehat{DA} < 360°.   \text{Why?}$$
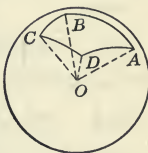
EXERCISES

1. Prove that any side of a spherical polygon is less than the sum of the remaining sides.

2. Show that any side of a convex spherical polygon is less than 180°.

3. Given a spherical $\triangle ABC$, with $AB = 57°$ and $AC = 125°$. Between what limits must BC lie?

4. Given a spherical quadrilateral three of whose sides are 77°, 98°, and 45° respectively. Between what limits must the fourth side lie? Between what limits must the fourth side lie if three sides are 27°, 33°, and 100° respectively?

841. Theorem. *Two triedral angles are congruent or symmetric if a face angle and the two adjacent diedral angles of one are equal respectively to a face angle and the two adjacent diedral angles of the other.*

Given triedral $\measuredangle V\text{-}ABC$ and $V'\text{-}DEF$, with $\angle AVC = \angle DV'F$, and the adjacent diedral angles equal.

To prove that the triedral angles are congruent or symmetric.

Proof. If parts are arranged in the same order, superpose as in the analogous case in plane geometry (§ 248). If parts are not in the same order, extend the edges through the vertex of one and superpose the other upon the symmetric triedral angle thus formed.   § 836
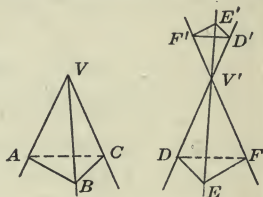
842. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if a side and the two adjacent angles of one are equal respectively to a side and the two adjacent angles of the other.*

Construct corresponding central triedral angles and apply the previous theorem.

843. Theorem. *Two isosceles symmetric spherical triangles are congruent.*

844. Theorem. *Two triedral angles are congruent or symmetric if two face angles and the included diedral angle of one equal respectively to two face angles and the included diedral angle of the other.*

Superpose if arranged in the same order. If arranged in opposite order, superpose one upon the triedral angle symmetric to the other.

845. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.*

Construct corresponding central triedral angles and apply the previous theorem.

846. Theorem. *On the same sphere or on equal spheres, two spherical triangles are congruent or symmetric if the three sides of one are equal respectively to the three sides of the other.*

Construct the corresponding central triedral angles and use § 610 when parts are in the same order. When parts are in the opposite order, construct a triedral angle symmetric to one by § 836, and then apply § 610.
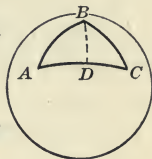
847. A triedral angle is **isosceles** if it has two equal face angles. It is equilateral if it has all its face angles equal.

848. Theorem. *If a spherical triangle is isosceles, the angles opposite the equal sides are equal.*

Given the isosceles spherical $\triangle ABC$, with $\overset{\frown}{AB} = \overset{\frown}{CB}$.

To prove $\angle A = \angle C$.

Proof. Draw the great circle arc BD bisecting $\overset{\frown}{AC}$.

Then spherical $\triangle ABD$ and CBD are congruent or symmetric.

§ 846

$$\therefore \angle A = \angle C. \qquad \text{Why?}$$

EXERCISES

1. If a triedral angle is isosceles, the diedral angles opposite the equal face angles are equal.

2. If a spherical triangle is equilateral, it is equiangular.

3. If a triedral angle is equilateral, all the diedral angles are equal.

4. State and prove theorems on the sphere corresponding to the following theorems in plane geometry:

(1) All points in the perpendicular bisector of a straight line are equidistant from the extremities of the line. § 205

(2) All points equidistant from the extremities of a straight line lie in the perpendicular bisector of the line. § 206

(3) All points in the bisector of an angle are equidistant from the sides of the angle. § 199

5. In the isosceles triangle of § 848, could $\overset{\frown}{AB}$ and $\overset{\frown}{CB}$ each be greater than 90°? Could each be greater than 180°? Could $\angle ABC$ be 20° and $\overset{\frown}{AC}$ be 5°? Could $\angle ABC$ be 20° and $\overset{\frown}{AC}$ be 20°? Illustrate on a globe.

849. A triedral angle is called a **rectangular,** a **birectangular,** or a **trirectangular** triedral angle, according as it has one, two, or three right diedral angles.

Three concurrent faces of a rectangular solid form a trirectangular triedral angle. Three concurrent faces of a right prism with an oblique triangular base form a birectangular triedral angle.

850. A spherical triangle is called a **right,** a **birectangular,** or a **trirectangular** spherical triangle, according as it has one, two, or three right spherical angles.

851. The following are derived from §§ 779 and 785:

(1) *In a birectangular spherical triangle, the sides opposite the equal angles are quadrants, and conversely.*
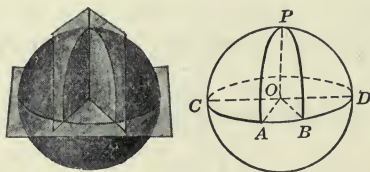
In the figure, O is a birectangular triedral angle, and APB is a birectangular spherical triangle. The faces, or arcs, that pass through P form right angles with the other face, or arc.

(2) *In a trirectangular spherical triangle, the sides are quadrants, and conversely.*
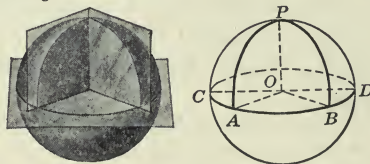
In the figure, O is a trirectangular triedral angle, and APB is a trirectangular spherical triangle. Here each vertex is the pole of the opposite side of the spherical triangle.   § 785

852. Theorem. *Three mutually perpendicular planes passed through the center of a sphere divide the sphere into eight congruent trirectangular spherical triangles.*

EXERCISES

1. Find the area of a trirectangular spherical triangle on a sphere having a radius of 10 in.

2. Find the area of a birectangular spherical triangle with a vertex angle of 60°, on a sphere 40 ft. in diameter.

3. What are the volumes cut out of the sphere by the corresponding central triedral angles in Exercises 1 and 2?

4. On a sphere having a radius of 12 in. two sides of an isosceles spherical triangle are quadrants. Find the length of the third side if the vertex angle is 60°.

5. On a sphere having a radius of 4 in. the sides of a spherical triangle are 80°, 140°, and 120° respectively. Find the length of each side.

6. Find the area of a birectangular spherical triangle having one angle 1°, on a sphere of radius r.

7. What is the volume cut out of the sphere by the central triedral angle corresponding to the spherical triangle described in Exercise 6?

POLAR TRIANGLES

853. If from the vertices of the spherical triangle ABC as poles, great circles are drawn, the sphere is divided into eight spherical triangles. If the intersection of the circles having A and B as poles, that is on the same side of AB as C, is labeled C', and similarly for the vertices A' and B'; then the spherical triangle $A'B'C'$ is the **polar triangle** of ABC.
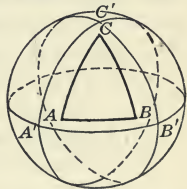
"On the same side of" means that C and C' are both in one of the hemispheres formed by the great circle of which AB is an arc.

Two spherical triangles related as ABC and $A'B'C'$ are spoken of as **polar triangles.**

Other spherical triangles should be constructed, some having sides of few degrees and others having sides near 180°. The polar triangles of these should be drawn and labeled as directed.
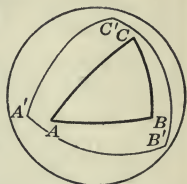
It is important that these relations should be clearly understood, as they are used in proving many theorems.

854. Theorem. *If one spherical triangle is the polar of another, then the second is the polar of the first.*

Given the spherical $\triangle ABC$ and its polar $\triangle A'B'C'$.

To prove that $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

Proof. A is the pole of $\overset{\frown}{B'C'}$, and C is the pole of $\overset{\frown}{A'B'}$.   Given

Then B is at a quadrant's distance from both A and C. Why?

Hence B' is the pole of AC.   § 787

Similarly A' is the pole of $\overset{\frown}{BC}$, and C' the pole of $\overset{\frown}{AB}$.

$\therefore \triangle ABC$ is the polar triangle of $\triangle A'B'C'$.   § 853

EXERCISES

1. One side of a spherical triangle on the earth's surface is on the equator. Prove that a vertex of the polar triangle is either at the north pole or at the south pole of the earth.

2. Prove the theorem of § 854 by taking as given that $\triangle ABC$ is the polar triangle of $\triangle A'B'C'$.

3. If a vertex of a spherical triangle is at the pole of the great circle of which one side is an arc, prove that one vertex of the polar triangle is at this pole and that one side is on this great circle.

4. Draw the following spherical triangles and their polar triangles:

(1) A triangle each side of which is less than a quadrant.

(2) A triangle each side of which is greater than a quadrant.

(3) A triangle only one side of which is greater than a quadrant.

(4) A birectangular triangle.

(5) A trirectangular triangle.

Note. A slated sphere should be used in making these constructions. A pair of compasses should be so adjusted that they will always strike an arc of a great circle on the sphere being used.

5. Can the sides of a spherical triangle intersect the sides of its polar triangle? Under what conditions?

6. Prove the theorem of § 854 by using this figure.
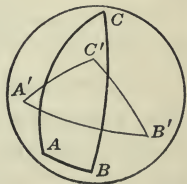
7. Prove that a trirectangular triangle is its own polar triangle.

855. Theorem. *In two polar triangles, each angle of the one is equal in degrees to the supplement of the side opposite to it in the other.*
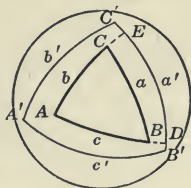

Given the polar triangles ABC and $A'B'C'$, with the letter at each vertex of an angle denoting its value in degrees, and the letters a, b, c and a', b', c' denoting the values of the opposite sides in degrees.

To prove that $A + a' = 180°$, $B + b' = 180°$, $C + c' = 180°$;
and $\qquad\qquad A' + a = 180°$, $B' + b = 180°$, $C' + c = 180°$.

Proof. Let D and E respectively be the intersections of $\widehat{AB}$ and $\widehat{AC}$, produced if necessary, with $\widehat{B'C'}$.

Then $\qquad\qquad \widehat{C'D} = 90°$, and $\widehat{EB'} = 90°$. $\qquad\qquad$ § 785

Or $\qquad \widehat{C'E} + \widehat{ED} = 90°$, and $\widehat{ED} + \widehat{DB'} = 90°$. $\qquad$ Why?

Then $\qquad \widehat{C'E} + \widehat{ED} + \widehat{ED} + \widehat{DB'} = 180°$. $\qquad$ Why?

But $\qquad \widehat{C'E} + \widehat{ED} + \widehat{DB'} = a'$, and $\widehat{ED} = °\angle A$. §§ 109, 829

$\qquad\qquad\qquad \therefore A + a' = 180°$. $\qquad\qquad\qquad$ Why?

Similarly $B + b' = 180°$, and $C + c' = 180°$.

Complete the proof by starting with $\triangle A'B'C'$.

EXERCISES

1. If the sides of a spherical triangle have respectively 70°, 80°, and 110°, find the values of the angles of its polar triangle.

2. Could the sides of a spherical triangle be each 8°? What would then be the value of the angles in the polar triangle?

3. Could the angles of a spherical triangle be 8°, 9°, and 10° respectively? What would then be the sides of the polar triangle?

4. Find the area of the triangle that is the polar of a birectangular spherical triangle having an angle of 1°, on a sphere 4 ft. in radius.

856. Theorem. *If two spherical triangles on the same sphere or on equal spheres are mutually equiangular, they are mutually equilateral, and are either congruent or symmetric.*
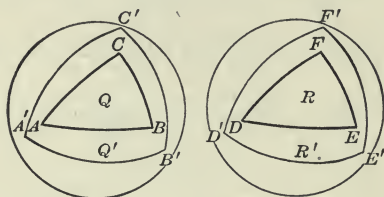
Given two mutually equiangular spherical triangles Q and R on equal spheres.

To prove that Q and R are mutually equilateral, and are either congruent or symmetric.

Proof. Construct △ Q' and R', the polar triangles of △ Q and R respectively.

△ Q and R are mutually equiangular.	Given
Then △ Q' and R' are mutually equilateral.	§ 855
Hence △ Q' and R' are either congruent or symmetric.	§ 846
And △ Q' and R' are mutually equiangular.	Why?
But △ Q and R are the polar triangles of △ Q' and R'.	§ 854
Then △ Q and R are mutually equilateral.	§ 855
∴ △ Q and R are either congruent or symmetric.	§ 846

857. Theorem. *If two triedral angles have their diedral angles respectively equal, their face angles are respectively equal, and they are either congruent or symmetric.*

EXERCISES

1. Is there a theorem in plane geometry analogous to the theorems of §§ 856, 857?

2. If two angles of a spherical triangle are equal, the triangle is isosceles.

3. If a spherical triangle is equiangular it is equilateral.

4. If two angles of a spherical triangle are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle; and conversely. Proof of theorem is similar to that of § 184.

858. Theorem. *The sum of the angles of a spherical triangle is greater than 180° and less than 540°.*
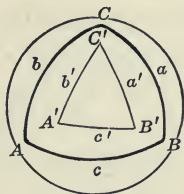
Given the spherical $\triangle ABC$ with the letter at each vertex of an angle denoting its value in degrees, and the letters a, b, c denoting the values of the opposite sides in degrees.

To prove $A+B+C>180°$ and $<540°$.

Proof. Construct the polar $\triangle A'B'C'$ of $\triangle ABC$.

Then	$A+a'=180°$, $B+b'=180°$, $C+c'=180°$.	§ 855
Hence	$A+B+C+a'+b'+c'=540°$.	§ 105
But	$a'+b'+c'<360°$.	§ 840
	$\therefore A+B+C>180°$.	§ 178
Also	$a'+b'+c'>0°$.	Why?
	$\therefore A+B+C<540°$.	Why?

859. Theorem. *The sum of the diedral angles of a triedral angle is greater than 180° and less than 540°.*

EXERCISES

1. How would you draw a triangle on a sphere, the sum of the angles of which would be only a little greater than 180°? Only a little less than 540°? Could you draw such triangles on a small sphere or would it be necessary to have a large sphere?

2. What is the locus of points on a sphere and equally distant from two intersecting arcs of great circles of the sphere?

3. What is the locus of the points on a sphere that are equally distant from the ends of an arc of a great circle?

4. If the sides of a spherical triangle each have less than 3°, what can be said of the angles of its polar triangle? Could the angles of a spherical triangle be respectively 170°, 6°, and 12°?

AREAS OF SPHERICAL POLYGONS

860. Theorem. *On the same sphere or on equal spheres two symmetric spherical triangles are equal in area.*
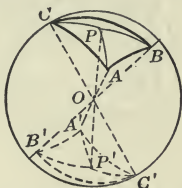
Given the two symmetric spherical triangles ABC and $A'B'C'$.

To prove that $\triangle ABC = \triangle A'B'C'$.

Proof. Place the two triangles so that they are formed by the same pyramidal surface with its vertex at O. § 838

Let P be the pole of a small circle passing through A, B, and C.

Draw the diameter POP' and the great circle arcs PA, PB, PC, $P'A'$, $P'B'$, and $P'C'$.

Then $\qquad \widehat{PA} = \widehat{PB} = \widehat{PC}$. § 783

But $\qquad \widehat{P'A'} = \widehat{PA}, \widehat{P'B'} = \widehat{PB}, \widehat{P'C'} = \widehat{PC}$. Why?

Then $\qquad \widehat{P'A'} = \widehat{P'B'} = \widehat{P'C'}$. Why?

And $\triangle PAB$ and $P'A'B'$ are isosceles as well as symmetric.

Hence $\qquad \triangle PAB \cong \triangle P'A'B'$. § 843

Similarly $\triangle PBC \cong \triangle P'B'C'$, and $\triangle PCA \cong \triangle P'C'A'$.

Then $\qquad \triangle PAB + \triangle PBC + \triangle PCA = \triangle P'A'B' +$
$$\triangle P'B'C' + \triangle P'C'A'. \qquad \text{§ 105}$$
$$\therefore \triangle ABC = \triangle A'B'C'. \qquad \text{§§ 109, 104}$$

EXERCISES

1. Draw a figure in which the pole P, § 860, falls outside the triangle ABC. What difference would this make in the proof?

2. A sphere can be divided into four, eight, or twenty equal equilateral spherical triangles.

861. Spherical Excess. The excess of the sum of the angles of a spherical triangle over 180° is called the **spherical excess of the triangle.**

The **spherical excess of a spherical polygon** of n sides is the excess of the sum of its angles over $(n-2)180°$, that is, it is the excess over the sum of the angles of a plane polygon of the same number of sides.

862. Area of a Spherical Triangle. The sides as well as the angles of a spherical triangle are usually given in degrees. It is evident then that the number of square units in the area of a spherical triangle will depend upon the radius of the sphere on which the triangle is. When the radius of the sphere is not given it is convenient to have a unit of measure that is some fractional part of the surface of a sphere.

863. Spherical Degree. One half the area of a lune of 1°, or the area of a birectangular triangle having a vertex angle of 1°, or $\frac{1}{720}$ of the surface of a sphere, is called a **spherical degree.** It is the unit of measure of spherical polygons.

A polyedral angle whose vertex is at the center of a sphere cuts a spherical polygon out of the sphere. The area of this polygon in spherical degrees is called the measure of the polyedral angle. In this manner the magnitudes of polyedral angles may be compared.
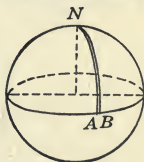
EXERCISES

1. How many spherical degrees in a lune of 20°? Of 46° 30′?

2. How many spherical degrees in a trirectangular spherical triangle? What is the area in square inches of a trirectangular triangle on a sphere having a radius of 10 in.?

3. How many square feet in the area of a birectangular triangle with vertex angle 18° on a sphere 6 ft. in. diameter?

4. Find the spherical excess in spherical triangles having the following angles: (1) 100°, 117°, 136°. (2) 141°, 97° 30′, 67° 45′. (3) 116° 23′ 14″, 127° 43′ 18″, 69° 45′ 15″.

5. Find the spherical excess in a spherical pentagon having angles of 113°, 97°, 119° 17′, 141° 23′, and 172° 27′,

864. Theorem. *A spherical triangle is equivalent to a lune on the same sphere, whose angle is half the spherical excess of the triangle.*
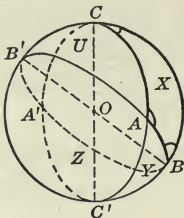
Given the spherical $\triangle ABC$ on a sphere of surface S and center O, with A, B, and C denoting the value in degrees of the angles at the respective vertices.

To prove that $\triangle ABC =$ a lune of angle $\frac{1}{2}(A+B+C-180°)$.

Proof. Complete the great circles of the sides of $\triangle ABC$, and let one of them as BC from the boundary of the hemisphere which is divided into four triangles, ABC, or X, ABC' or Y, $AB'C'$ or Z, and $AB'C$ or U.

	$\triangle BA'C$ on the reverse hemisphere $= \triangle Z$.	§ 860
	Lune of $\angle A = \triangle X + \triangle Z$.	Why?
	Lune of $\angle B = \triangle X + \triangle U$.	Why?
	Lune of $\angle C = \triangle X + \triangle Y$.	Why?
Then	lune of $\angle (A+B+C) = 3\triangle X + \triangle Z + \triangle U + \triangle Y$.	§ 105
But	$\triangle X + \triangle Z + \triangle U + \triangle Y = \frac{1}{2}S =$ lune of $\angle 180°$.	Why?
Then	lune of $\angle (A+B+C) = 2\triangle X +$ lune of $\angle 180°$.	§ 111
Or	$2\triangle X =$ lune of $\angle (A+B+C-180°)$.	Why?
	$\therefore \triangle ABC =$ lune of $\angle \frac{1}{2}(A+B+C-180°)$.	Why?

865. Theorem. *On the same sphere or on equal spheres, the areas of two spherical triangles are to each other as their spherical excesses.*

866. Theorem. *If A denotes the area of a spherical triangle, E the number of degrees in its spherical excess, and r the radius of the sphere, then the area is given by the formula:* $A = \dfrac{\pi r^2 E}{180}$.

867. Theorem. *A spherical polygon is equivalent to a lune on the same sphere, whose angle is half the spherical excess of the polygon.*

Divide the spherical polygon into spherical triangles by drawing the diagonals from one vertex. Then the area of the polygon is equivalent to the sum of the areas of the triangles, and the spherical excess of the polygon is equal to the sum of the excesses of the triangles.
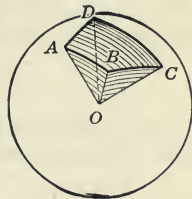
868. Theorem. *The area of a spherical polygon is given by the formula* $A = \dfrac{\pi r^2 E}{180}$, *where r is the radius of the sphere and E the number of degrees in the spherical excess.*

Note. This theorem was stated by A. Girard in 1629 and was first completely proved by Cavalieri a few years later.

869. Spherical Pyramid. The portion of the volume of a sphere bounded by a spherical polygon and the planes of its sides is called a **spherical pyramid**.

The polygon is the **base** of the spherical pyramid, and the center of the sphere is its **vertex**.

Thus, $O\text{-}ABCD$ is a spherical pyramid.

870. Theorem. *The volume of a spherical pyramid is equal to one-third the product of the area of its base by the radius of the sphere.*

EXERCISES

1. Find the areas of triangles on spheres of the given radii, and having angles as follows:

(1) 80°, 140°, 120°, $r = 10$ in.   (3) 125°, 135°, 145°, $r = 50$ ft.
(2) 150°, 75°, 110°, $r = 16$ in.   (4) 179°, 179°, 179°, $r = 10$ ft.
  (5) 112° 30′ 15″, 127° 43′ 30″, 175° 27′ 15″, $r = 20$ ft.
  (6) 37° 43′ 27″, 15° 45′ 34″, 170° 45′ 43″, $r = 4000$ mi.

2. Find the areas of polygons on spheres of the given radii, and having angles as follows:

(1) 70°, 170°, 100°, 125°, $r = 12$ in.
(2) 100°, 98°, 175°, 160°, 128°, 96°, $r = 8$ ft.
(3) 75° 45′ 17″, 127° 14′ 16″, 102° 43′ 16″, 150° 43′ 41″, $r = 20$ in.

3. Find the area of the triangle on the earth's surface, determined by the meridians at 48° and 75° west longitude, and the equator. Use 4000 miles for the radius of the earth.

4. Find the volume of a spherical pyramid, given that its base is an equiangular triangle having each angle 120°, on a sphere of radius 9 in.

5. Find the volume of a spherical pyramid, given that its base is a spherical hexagon each angle of which is 130°, on a sphere of 12 in. radius.

6. The volume of a spherical pyramid whose base is an equiangular spherical triangle with angles of 105° is 256π cu. in. Find the diameter of the sphere.

7. What is the greatest area that a triangle on the surface of the earth may have if the sum of its angles is not to differ by more than 1° from 180°? If the sum is not to differ by more than 1′ from 180°?

8. A triangle having an area of 62 sq. in. is on a sphere that has an area of 500 sq. in. Two of its angles are 92° and 135° respectively. Find the third angle.

9. The sides of a spherical triangle are 79°, 88°, and 115°, and it is on a sphere whose radius is 18 in. Find the area of its polar triangle.

QUESTIONS

1. Define a circle. A sphere. Is a circle an area? Is a sphere a solid? Define a sphere as a locus. Define a circle as a locus.

2. Is it in agreement with the definition of a sphere to speak of the area of a sphere? What is the volume of a sphere? Compare the definition of the area of a circle with that of the volume of a sphere.

3. What is a spherical angle? A spherical triangle? Is a triangle an area?

4. How is a plane angle measured? A spherical angle? A diedral angle? A polyedral angle?

5. What is a lune? A wedge? How may lunes be compared in size? How may wedges be compared in size?

6. Define the sector of a sphere, and compare with the sector of a circle.

7. Define the segment of a sphere, and compare with the segment of a circle. Do we speak of the segment of a circle of one base? Is there any reason for not defining the segment of a circle so that we can speak of a segment of two bases?

8. Is the distance between two points on a sphere defined so that it is analogous to the distance between two points in a plane?

9. Explain why it is well to treat spherical polygons in connection with polyedral angles. State the relations between the parts of a spherical triangle and the parts of the corresponding polyedral angle.

10. State several theorems about spherical triangles that are analogous to theorems about plane triangles. Is a plane triangle determined when its angles are known? Is a spherical triangle determined when its angles are known?

11. What is a polar triangle? What are some relations between a spherical triangle and its polar triangle? State a striking difference between the angles of a spherical triangle and those of a plane triangle.

12. How many points determine a circle? A sphere?

13. How many points on a sphere determine a small circle? A great circle?

14. Define axis, pole, polar distance.

15. If a great circle is perpendicular to a small circle, what must it contain? Is the converse of this true?

16. Give an outline of the steps in determining the formulas for the area of a sphere. State these formulas, and state when each is the best one to use.

17. Do the same for the volume of a sphere.

18. What is the formula for the area of a zone? A lune? A spherical triangle? Can you give a practical application of the use of each of these areas?

19. What is the formula for the volume of a spherical segment? A wedge? A spherical pyramid? A spherical sector? Can you give a practical application of the use of each of these volumes?

20. What remarkable relation is there between the volumes of a cylinder, sphere, and cone? Is there a similar relation between their areas?

GENERAL EXERCISES

1. The volume of any polyedron circumscribed about a sphere is equal to the area of its surface times one-third the radius of the sphere.

2. The volumes of polyedrons circumscribed about the same sphere are to each other as their surfaces.

3. What is the locus of all points in space equally distant from two given points and at a distance r from a third given point?

4. Show how to find the pole of a given circle on a sphere.

5. Construct a circle through three given points on a sphere.

6. Four balls of equal radii r are placed on a plane and touching each other in such a way that their centers form a square. A fifth ball of the same radius is placed upon the four first balls. Find the distance of its center from the plane.

7. A cylinder is inscribed in a sphere of radius r and has a lateral area equal to one-half the area of the sphere. Find the radius of the cylinder.   *Ans.* $\frac{1}{2}r\sqrt{2}$.

8. A zone of one base is a mean proportional between the remaining part of the area of the sphere and the total area of the sphere. How far is its base from the center of the sphere?

9. A spherical segment of one base is half as large as the spherical sector to which it belongs. If the radius of the sphere is 4 in., find the altitude of the segment.

10. The inside of a glass is in the form of a right circular cone whose vertex angle is 60° and the diameter of whose base is 4 in. The glass is filled with water and the largest sphere that can be immersed in the water is placed in the glass. Find the volume of water remaining in the glass.

11. An irregular portion, but less than half, of a material sphere, is given. Show how to construct the diameter of the sphere.

12. In the figure of § 813, the plane MNP is tangent to the sphere inscribed in the cube. If the edge of the cube is a find DN.

13. Two spheres whose radii are respectively 6 in. and 8 in. have their centers 10 in. apart. Find the volume of the portion common to the two spheres. This is the form of a spherical lens.
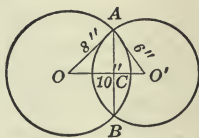
14. A hole of diameter 4 in. was bored through the center of a sphere of diameter 12 in. Find the volume of the part cut away.

15. A sphere is inscribed in a right circular cylinder having an altitude equal to the diameter of the sphere. Two planes parallel to the base of the cylinder cut the cylinder and the sphere. Prove that the surface of the cylinder lying between the planes equals the area of the zone between the planes.

16. A circle is circumscribed about an equilateral triangle. Find the ratios of the areas and the volumes of the solids formed by revolving the triangle and the circle about an altitude of the triangle as axis.

17. A hollow copper sphere used as a float in water weighs 10 oz., and has a diameter of 5 in. How heavy a weight will it support?
                                         Ans. Less than 27.9 oz.

FORMULAS FOR REFERENCE

Lateral area of prism. $S = pe$, $S = ph$.

Lateral area of circular cylinder. $S = pe$, $S = ch$.

Volume of rectangular parallelepiped. $V = abh$.

Volume of parallelepiped. $V = Bh$.

Volume of prism or cylinder. $V = Bh$.

Volume of hollow circular cylinder. $V = \frac{1}{4}\pi h (D+d)(D-d)$.

Lateral area of regular pyramid. $S = \frac{1}{2} ps$.

Lateral area of right circular cone. $S = \frac{1}{2} cs = \pi rs$.

Total area of right circular cone. $T = \pi rs + \pi r^2 = \pi r(s+r)$.

Lateral area of frustum of regular pyramid. $S = \frac{1}{2} s(P+p)$.

Lateral area of frustum of right circular cone. $S = \pi(R+r)s$.

Volume of pyramid or cone. $V = \frac{1}{3} Bh$.

Volume of circular cone. $V = \frac{1}{3}\pi r^2 h$.

Volume of frustum of pyramid or cone.
$$V = \frac{1}{3} h (B_1 + B_2 + \sqrt{B_1 B_2}).$$

Volume of frustum of circular cone.
$$V = \frac{1}{3}\pi h (r_1^2 + r_2^2 + r_1 r_2) = \frac{1}{12}\pi h (d_1^2 + d_2^2 + d_1 d_2).$$

Volume of prismatoid. $V = \frac{1}{6} h (B_1 + B_2 + 4M)$.

Area of a sphere. $S = 4\pi r^2$.

Area of a zone. $Z = 2\pi rh$.

Area of a lune. $L = \dfrac{u}{360}(4\pi r^2)$.

Volume of a sphere. $V = \frac{4}{3}\pi r^3 = \frac{1}{6}\pi d^3 = \frac{1}{3} rS$.

Volume of spherical segment. $V = \frac{1}{2}\pi h (r_1^2 + r_2^2) + \frac{1}{6}\pi h^3$.

Volume of spherical wedge. $V = \dfrac{u}{360}(\frac{4}{3}\pi r^3)$.

Area of spherical polygon. $A = \dfrac{\pi r^2 E}{180}$.

USEFUL NUMBERS

1 cu. ft. of water weighs 62.5 lb. (approx.) = 1000 oz.

1 gal. of water weighs $8\frac{1}{3}$ lb. (approx.).

1 atmosphere pressure = 14.7 lb. per sq. in. = 2116. lb. per sq. ft

1 atmosphere pressure = 760 mm. of mercury.

A column of water 2.3 ft. high = a pressure of 1 lb. per sq. in.

1 Kg. = 2.2 lb. (approx.).

1 gal. = 231 cu. in. (by law of Congress).

1 cu. ft. = $7\frac{1}{2}$ gal. (approx.) or better 7.48 gal.

1 cu. ft. = 4 bu. (approx.).

1 bbl. = 4.211 − cu. ft. (approx.).

1 bu. = 2150.42 cu. in. (by law of Congress) = 1.24446 − cu. ft.

1 bu. = $\frac{5}{4}$ cu. ft. (approx.).

1 perch = $24\frac{3}{4}$ cu. ft. but usually taken 25 cu. ft.

1 in. = 25.4 mm. (approx.).

1 m. = 39.37 in. (by law of Congress).

1 lb. (avoirdupois) = 7000 grains (by law of Congress).

1 lb. (troy or apothecaries) = 5760 grains.

$\pi = 3.14159265358979+ = 3.1416 = \frac{355}{113} = 3\frac{1}{7}$ (all approx.).

$\sqrt{2} = 1.4142136.$ $\sqrt{3} = 1.7320508.$

$\sqrt{5} = 2.2360680.$ $\sqrt{6} = 2.4494897.$

$\sqrt[3]{2} = 1.2599210.$ $\sqrt[3]{3} = 1.4422496.$

SYLLABUS OF PLANE GEOMETRY

(FOR REFERENCE OR REVIEW)

Chapter I. Straight-Line Figures

PRELIMINARY PROPOSITIONS AND AXIOMS

§ 19. **Experiment.** If two triangles have two angles and the included side of one equal respectively to two angles and the included side of the other, the triangles are equal.

§ 21. In equal triangles, the corresponding sides are equal and the corresponding angles are equal.

§ 23. **Experiment.** If two triangles have two sides and the included angle of one equal respectively to two sides and the included angle of the other, the triangles are equal.

§ 25. **Construction.** To construct a triangle when the three sides are given.

§ 27. **Experiment.** If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are equal.

§ 30. **Experiment.** The sum of the angles of a triangle is equal to 180°.

§ 32. If two triangles have two angles of one equal respectively to two angles of the other, the third angles are equal.

§ 35. **Experiment.** If one straight line intersects another straight line, the vertical angles are equal.

§ 49. If two adjacent angles have their exterior sides in the same straight line, they are supplementary.

§ 51. If two adjacent angles are supplementary, their exterior sides are in the same straight line.

§ 53. Complements of the same angle or of equal angles are equal.

§ 55. Supplements of the same angle or of equal angles are equal.

§ 58. If one of the angles formed by two intersecting lines is a right angle, the other three angles also are right angles.

§ 59. The sum of all the angles about a point is equal to 360°, or four right angles.

§ 60. The sum of all the angles about a point, on the same side of a straight line passing through the point, is equal to 180°.

§ 61. All right angles are equal.

§ 62. All straight angles are equal.

§ 63. A right angle is half a straight angle.

§ 65. One straight line, and only one, can be drawn from one point to another.

§ 66. Two points determine a straight line.

§ 67. A straight line may be produced indefinitely in both directions.

§ 68. Any number of straight lines can be drawn through a given point.

§ 70. Two straight lines can intersect in only one point.

§ 71. Two intersecting lines determine a point.

§ 74. At a given point in a given straight line, only one perpendicular can be drawn to that line.

§ 76. Only one perpendicular can be drawn to a given straight line from a given external point.

§ 77. Through a given point, one straight line, and only one, can be drawn perpendicular to a given straight line.

TRIANGLES

§ 84. **Problem.** To construct an equilateral triangle when a side is given.

§ 85. **Problem.** To construct an isosceles triangle when the base and one of the equal sides are given.

§ 86. **Theorem.** In an isosceles triangle, the angles opposite the equal sides are equal.

§ 87. **Theorem.** An equilateral triangle is also equiangular.

§ 88. **Theorem.** Each angle of an equilateral triangle is equal to 60°.

§ 89. **Theorem.** If two angles of a triangle are equal the sides opposite the equal angles are equal and the triangle is isosceles.

§ 90. **Theorem.** An equiangular triangle is also equilateral.

§ 91. **Theorem.** The sum of the acute angles of a right triangle is equal to 90°, or one right angle.

§ 92. **Theorem.** If two right triangles have one acute angle of one equal to one acute angle of the other, the other acute angles, also, are equal.

§ 93. **Theorem.** Each acute angle of an isosceles right triangle is equal to 45°.

§ 97. If two right triangles have a side and an acute angle of one equal respectively to the corresponding side and the corresponding acute angle of the other, the triangles are equal.

§ 98. **Theorem.** If two triangles have a side and any two angles of one equal respectively to the corresponding side and the two corresponding angles of the other, the triangles are equal.

§ 101. If two right triangles have two sides of one equal respectively to two corresponding sides of the other, the triangles are equal.

AXIOMS OF EQUALITY

§ 104. Axiom. Quantities which are equal to the same quantity or to equal quantities are equal to each other.

§ 105. Axiom. If equals are added to equals, the sums are equal.

§ 106. Axiom. If equals are subtracted from equals, the remainders are equal.

§ 107. Axiom. If equals are multiplied by equals, the products are equal.

§ 108. Axiom. If equals are divided by equals, the quotients are equal. The divisor must not be zero.

§ 109. Axiom. The whole is equal to the sum of all its parts.

§ 110. Axiom. The whole is greater than any of its parts.

§ 111. Axiom. A quantity may be substituted for its equal in any operation.

§ 113. Theorem. If two triangles have the three sides of one equal respectively to the three sides of the other, the triangles are equal.

PARALLEL LINES

§ 115. Theorem. Straight lines in the same plane which are perpendicular to the same straight line can never meet.

§ 117. Theorem. Straight lines in the same plane which are perpendicular to the same straight line are parallel.

§ 118. Axiom. Through a given point only one straight line can be drawn parallel to another straight line.

§ 119. Axiom. Straight lines in the same plane which are not parallel will meet if sufficiently prolonged.

§ 122. Experiment. If two parallel lines are cut by a transversal:
    (1) The alternate interior angles are equal.
    (2) The alternate exterior angles are equal.
    (3) The exterior interior angles are equal.
    (4) The consecutive interior angles are supplementary.
    (5) The consecutive exterior angles are supplementary.

§ 123. Theorem. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

§ 124. Theorem. Two straight lines parallel to a third straight line are parallel to each other.

§ 127. Theorem. If two straight lines in the same plane are cut by a transversal, making the alternate interior angles equal, the lines are parallel.

§ 128. Theorem. If two straight lines in the same plane are cut by a transversal, making the exterior interior angles equal, the lines are parallel.

§ 129. Theorem. If two straight lines in the same plane are cut by a transversal, making the alternate exterior angles equal, the lines are parallel.

§ 130. Theorem. If two straight lines in the same plane are cut by a transversal, making the consecutive exterior angles supplementary, the lines are parallel.

§ 131. Theorem. If two straight lines in the same plane are cut by a transversal, making the consecutive interior angles supplementary, the lines are parallel.

§ 132. Theorem. If two parallel lines are cut by a transversal, the alternate interior angles are equal.

§ 133. Theorem. If two parallel lines are cut by a transversal, the exterior interior angles are equal.

§ 134. Theorem. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

§ 135. Theorem. If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.

§ 136. Theorem. If two parallel lines are cut by a transversal, the consecutive exterior angles are supplementary.

§ 138. Theorem. If two angles have their sides parallel, right side to right side, and left side to left side, the angles are equal.

§ 139. Theorem. If two angles have their sides parallel, right side to left side, and left side to right side, the angles are supplementary.

§ 140. Theorem. If two angles have their sides perpendicular, right side to right side, and left side to left side, the angles are equal.

§ 141. Theorem. If two angles have their sides perpendicular, right side to left side and left side to right side, the angles are supplementary.

§ 143. Theorem. The sum of the angles of a triangle is equal to 180°, or two right angles.

§ 144. Theorem. An exterior angle of a triangle is equal to the sum of the opposite interior angles, and is therefore greater than either of them.

§ 145. Theorem. In a triangle, there can be only one right angle, or one obtuse angle.

QUADRILATERALS

§ 154. Theorem. A diagonal divides a parallelogram into two equal triangles.

§ 155. Theorem. In any parallelogram, the opposite sides are equal, and the opposite angles are equal.

§ 156. Theorem. The diagonals of a parallelogram bisect each other.

§ 157. Theorem. Any two consecutive angles of a parallelogram are supplementary.

§ **158.** **Theorem.** Parallels cut off by two parallel lines are equal.

§ **159.** **Theorem.** If the opposite sides of a quadrilateral are equal, the figure is a parallelogram.

§ **160.** **Theorem.** If two opposite sides of a quadrilateral are equal and parallel, the figure is a parallelogram.

§ **161.** **Theorem.** If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.

§ **162.** **Theorem.** The line joining the middle points of two opposite sides of a parallelogram is parallel to and equal to each of the other two sides.

§ **163.** **Theorem.** The diagonals of a rectangle are equal.

§ **164.** **Theorem.** If a parallelogram has equal diagonals, it is a rectangle.

§ **165.** **Theorem.** The diagonals of a square are perpendicular to each other, and bisect the angles of the square.

POLYGONS

§ **170.** **Theorem.** The sum of the interior angles of any polygon is equal to 180° multiplied by two less than the number of sides.

§ **171.** **Theorem.** In an equiangular polygon, each angle is equal to $\dfrac{(n-2)180°}{n}$, or $\dfrac{(n-2)2 \text{ rt.} \measuredangle}{n}$.

§ **172.** **Theorem.** The sum of the exterior angles of a polygon, formed by prolonging one side at each vertex, is 360°.

INEQUALITIES

§ **173.** **Axiom.** If equals are added to unequals, the sums are unequal in the same order.

§ **174.** **Axiom.** If equals are subtracted from unequals, the remainders are unequal in the same order.

§ **175.** **Axiom.** If unequals are multiplied by positive equals, the products are unequal in the same order.

§ **176.** **Axiom.** If unequals are divided by positive equals, the quotients are unequal in the same order.

§ **177.** **Axiom.** If unequals are added to unequals in the same order, the sums are unequal in that order.

§ **178.** **Axiom.** If unequals are subtracted from equals, the remainders are unequal in the reverse order.

§ **179.** **Axiom.** If the first of three quantities is greater than the second and the second is greater than the third, then the first is greater than the third.

§ 181. **Axiom.** A straight line is the shortest line that can be drawn from one point to another.

§ 182. **Theorem.** The sum of two sides of a triangle is greater than the third side. The difference between two sides of a triangle is less than the third side.

§ 183. **Theorem.** If two sides of a triangle are unequal, the angles opposite these sides are unequal, and the greater angle is opposite the greater side.

§ 184. **Theorem.** If two angles of a triangle are unequal, the sides opposite these angles are unequal, and the greater side is opposite the greater angle.

§ 185. **Theorem.** A perpendicular is the shortest line that can be drawn from a point to a straight line.

§ 186. **Theorem.** If two straight lines are drawn from a point in a perpendicular to a given line, cutting off on the line unequal lengths from the foot of the perpendicular, the more remote is the greater.

DISTANCE

§ 191. **Theorem.** Two parallel lines are everywhere equally distant.

§ 192. **Theorem.** If a line bisects one side of a triangle and is parallel to a second side, it bisects the third side.

§ 193. **Theorem.** The middle point of the hypotenuse of a right triangle is equidistant from the three vertices.

§ 194. **Theorem.** If one acute angle of a right triangle is twice the other, the hypotenuse is twice the shorter side.

§ 196. **Theorem.** The line joining the middle points of two sides of a triangle is parallel to the third side and is equal to one-half the third side.

§ 197. **Theorem.** If three or more parallel lines intercept equal parts on one transversal, they intercept equal parts on every transversal.

§ 198. **Theorem.** The median of a trapezoid is parallel to the bases and is equal to half the sum of the bases.

§ 199. **Theorem.** All points in the bisector of an angle are equidistant from the sides of the angle.

§ 200. **Theorem.** All points equidistant from the sides of an angle are in the bisector of the angle.

§ 205. **Theorem.** All points in the perpendicular bisector of a straight line are equidistant from the extremities of the line.

§ 206. **Theorem.** All points equidistant from the extremities of a straight line lie in the perpendicular bisector of the line.

§ 207. **Theorem.** Two points, each equidistant from the extremities of a straight line, determine the perpendicular bisector of the line.

OTHER THEOREMS

§ 209. Theorem. The bisector of the angles of a triangle pass through a common point, which is equidistant from the sides of the triangle.

§ 210. Theorem. The perpendicular bisectors of the three sides of a triangle pass through a common point, which is equidistant from the three vertices of the triangle.

§ 211. Theorem. The medians of a triangle pass through a common point, which is two-thirds of the length of each median from its vertex.

§ 213. Theorem. The three altitudes of a triangle pass through a common point.

§ 247. Theorem. Two triangles are congruent if two sides and the included angle of one are equal respectively to two sides and the included angle of the other.

§ 248. Theorem. Two triangles are congruent if two angles and the included side of one are equal respectively to two angles and the included side of the other.

§ 249. Theorem. Two parallelograms are congruent if they have two adjacent sides and the included angle of one equal respectively to two adjacent sides and the included angle of the other.

§ 251. Theorem. If two angles of a triangle are unequal, the sides opposite these angles are unequal, the greater side being opposite the greater angle.

§ 252. Theorem. Only one perpendicular can be drawn to a given line at a given point in the line.

§ 253. Theorem. Only one perpendicular can be drawn to a given line from a given external point.

§ 254. Theorem. If two parallel lines are cut by a transversal, the alternate interior angles are equal.

§ 255. Theorem. If two parallel lines are cut by a transversal, the exterior interior angles are equal.

§ 256. Theorem. If two parallel lines are cut by a transversal, the consecutive interior angles are supplementary.

§ 257. Theorem. If two parallel lines are cut by a transversal, the alternate exterior angles are equal.

§ 258. Theorem. If two triangles have two sides of one equal respectively to two sides of the other but the included angle of the first triangle greater than the included angle of the second, then the third side of the first is greater than the third side of the second.

§ 259. Theorem. If two triangles have two sides of one equal to two sides of the other but the third side of the first greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

Chapter II. The Circle

§ **272. Experiment.** Equal central angles in the same circle or in equal circles intercept equal arcs. Conversely: Equal arcs are intercepted by equal central angles.

§ **273.** Of two unequal central angles, in the same circle or in equal circles, the greater central angle intercepts the greater arc.

Conversely: Of two unequal arcs, the greater arc is intercepted by the greater central angle.

ARCS AND CHORDS

§ **275. Theorem.** In the same circle or in equal circles, equal chords subtend equal arcs. Conversely: Equal arcs are subtended by equal chords.

§ **276. Theorem.** A diameter of a circle is greater than any other chord.

§ **277. Theorem.** A diameter perpendicular to a chord bisects the chord and the arc subtended by it.

§ **278. Theorem.** A diameter bisects the circle.

§ **279. Theorem.** A diameter which bisects a chord is perpendicular to the chord.

§ **280. Theorem.** The perpendicular bisector of a chord passes through the center of the circle.

§ **281. Theorem.** In the same circle or in equal circles, equal chords are equidistant from the center. Conversely: Chords equidistant from the center are equal.

§ **282.** The distance from the center of a circle to any point without the circle is greater than the radius, and the distance from the center to any point within the circle is less than the radius.

§ **283.** A point is without a circle when its distance from the center is greater than the radius. Also, a point is within a circle when its distance from the center is less than the radius.

TANGENTS AND SECANTS

§ **284. Theorem.** A perpendicular to a radius at its outer extremity can touch the circle at only one point.

§ **287. Theorem.** A straight line perpendicular to a radius at the point where the radius meets the circle is tangent to the circle.

§ **288. Theorem.** The radius drawn to the point of tangency is perpendicular to the tangent.

§ **289. Theorem.** Parallel lines intercept equal arcs on a circle.

§ 290. **Theorem.** Tangents to a circle from an external point are equal, and make equal angles with the line joining that point with the center.

§ 295. **Theorem.** The line of centers of two intersecting circles is the perpendicular bisector of their common chord.

§ 296. **Theorem.** If two circles are tangent to each other, the line of centers passes through the point of contact.

ANGLE MEASUREMENT

§ 303. **Theorem.** The arcs formed by two perpendicular diameters are equal.

§ 305. **Experiment.** A central angle equals in degrees its intercepted arc.

§ 306. **Theorem.** An inscribed angle equals in degrees one-half its intercepted arc.

§ 307. **Theorem.** Angles inscribed in the same arc are equal.

§ 308. **Theorem.** An angle inscribed in a semicircle is a right angle.

§ 309. **Theorem.** An angle formed by two intersecting chords equals in degrees one-half the sum of the intercepted arcs.

§ 310. **Theorem.** An angle formed by a tangent and a chord equals in degrees one-half the intercepted arc.

§ 311. **Theorem.** An angle formed by two secants, two tangents, or a tangent and a secant, equals in degrees one-half the difference of the intercepted arcs.

PROBLEMS OF CONSTRUCTION

§ 312. **Problem.** To construct a perpendicular to a given straight line from a given external point.

§ 313. **Problem.** To construct a perpendicular to a given straight line at a given point in the line.

§ 314. **Problem.** To bisect a given straight line.

§ 315. **Problem.** To bisect a given arc.

§ 316. **Problem.** To circumscribe a circle about a given triangle.

§ 317. **Problem.** To construct a circle through three points which are not in the same straight line.

§ 318. **Theorem.** Three points not in the same straight line determine a circle.

§ 319. **Problem.** To bisect a given angle.

§ 320. **Problem.** From a given point in a given line, to draw a line making an angle equal to a given angle.

§ 321. **Problem.** Lines may be added, subtracted, and multiplied.

§ 322. **Problem.** A line may be divided into equal parts.

§ **323.** **Problem.** To construct a triangle when two angles and the included side are given.

§ **324.** **Problem.** To construct a triangle when two sides and the included angle are given.

§ **325.** **Problem.** To inscribe a circle in a given triangle.

§ **326.** **Problem.** To construct a tangent to a given circle.

§ **327.** **Problem.** With a given straight line as a chord, to construct an arc of a circle in which a given angle may be inscribed.

§ **329.** **Problem.** To construct a triangle when two sides and the angle opposite one of them are given.

Chapter III. Area of Polygons

MEASURING

§ **333.** **Axiom.** Every geometric magnitude has a numerical measure.

§ **346.** The area of a rectangle is equal to the product of its base and altitude.

§ **347.** **Theorem.** The area of a square equals the square of its side.

§ **348.** **Theorem.** The areas of two rectangles are to each other as the products of their bases and altitudes.

§ **349.** **Theorem.** The areas of two rectangles having equal bases are to each other as their altitudes.

§ **350.** **Theorem.** The areas of two rectangles having equal altitudes are to each other as their bases.

§ **351.** **Theorem.** Two rectangles having equal bases and equal altitudes are equal in area.

§ **354.** **Theorem.** Any parallelogram is equivalent to a rectangle that has its base and altitude equal respectively to the base and altitude of the parallelogram.

§ **355.** **Theorem.** The area of a parallelogram is equal to the product of its base and altitude.

§ **356.** **Theorem.** The altitude of a parallelogram is equal to its area divided by its base.

§ **357.** **Theorem.** The base of a parallelogram is equal to its area divided by its altitude.

§ **358.** **Theorem.** Parallelograms having equal bases and equal altitudes are equivalent.

§ **359.** **Theorem.** Two parallelograms having equal bases are to each other as their altitudes.

§ **360.** **Theorem.** Two parallelograms having equal altitudes are to each other as their bases.

§ 361. Theorem. Any two parallelograms are to each other as the products of their bases and their altitudes.

§ 362. Theorem. The area of a triangle is equal to half the product of its base and its altitude.

§ 363. Theorem. The area of a triangle equals half the area of a parallelogram having the same base and altitude as the triangle.

§ 364. Theorem. The base of a triangle equals twice its area divided by its altitude.

§ 365. Theorem. The altitude of a triangle equals twice its area divided by its base.

§ 366. Theorem. Triangles having equal bases and equal altitudes are equivalent.

§ 367. Theorem. Two triangles having equal bases are to each other as their altitudes.

§ 368. Theorem. Two triangles having equal altitudes are to each other as their bases.

§ 369. Theorem. Any two triangles are to each other as the products of their bases and their altitudes.

§ 370. Theorem. The area of a trapezoid is equal to half the product of its altitude and the sum of its bases.

§ 371. Theorem. The area of a trapezoid equals the product of its altitude and its median.

§ 375. Theorem. Two triangles that have an angle of one equal to an angle of the other are to each other as the products of the sides including the equal angles.

§ 376. Theorem. The square constructed on the hypotenuse of a right triangle is equivalent to the sum of the squares constructed on the other two sides.

§ 377. Theorem. The square on either side of a right triangle is equivalent to the square on the hypotenuse minus the square on the other side.

PROJECTION

§ 380. Theorem. In any obtuse triangle, the square of the side opposite the obtuse angle is equivalent to the sum of the squares of the other two sides, plus twice the product of one of those sides and the projection of the other side upon it.

§ 381. Theorem. In any triangle, the square of a side opposite an acute angle is equivalent to the sum of the squares of the other two sides, minus twice the product of one of these sides and the projection of the other side upon it.

§ 382. Theorem. An angle of a triangle is acute, right, or obtuse according as the square of the opposite side is less than, equal to, or greater than the sum of the squares of the other two sides.

§ 383. Theorem. If any median of a triangle be drawn to a side:

(1) The sum of the squares of the other two sides is equal to twice the square of the median plus twice the square of half the bisected side.

(2) The difference of the squares of the other two sides is equal to twice the product of the bisected side and the projection of the median upon that side.

TRANSFORMATIONS AND CONSTRUCTIONS

§ 385. Problem. To construct a square that is equivalent to the sum of two given squares.

§ 386. Problem. To construct a square that is equivalent to the difference of two given squares.

§ 388. Problem. To transform a given quadrilateral into an equivalent triangle.

§ 389. Problem. To transform a given polygon into an equivalent triangle.

§ 390. Problem. To construct a square equivalent to a given rectangle or parallelogram.

§ 391. Problem. To construct a square equivalent to a given triangle.

§ 392. Problem. To construct a square equivalent to a given polygon.

Chapter IV. Proportion and Similarity

THEOREMS OF PROPORTION

§ 398. Theorem. If four numbers form a proportion, the product of the extremes equals the product of the means.

§ 399. Theorem. The mean proportional between two numbers equals the square root of their product.

§ 400. Theorem. If the two antecedents of a proportion are equal, the two consequents are equal.

§ 401. Theorem. If three terms of one proportion are equal respectively to three corresponding terms of another proportion, the fourth terms are equal.

§ 402. Theorem. If the product of two numbers equals the product of two other numbers, either two may be made the means and the other two the extremes of a proportion.

§ 403. Theorem. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

§ **404.** **Theorem.** If four numbers form a proportion, they are in proportion by inversion; that is, the second term is to the first as the fourth is to the third.

§ **405.** **Theorem.** If four numbers form a proportion, they are in proportion by alternation; that is, the first term is to the third as the second is to the fourth.

§ **406.** **Theorem.** If four numbers form a proportion, they are in proportion by addition; that is, the sum of the first two terms is to the second as the sum of the last two terms is to the fourth.

§ **407.** **Theorem.** If four numbers form a proportion, they are in proportion by subtraction; that is, the difference of the first two terms is to the second as the difference of the last two terms is to the fourth.

§ **408.** **Theorem.** If four numbers form a proportion, they are in proportion by addition and subtraction; that is, the sum of the first two is to their difference as the sum of the last two is to their difference.

PROPORTIONAL LINES

§ **411.** **Theorem.** A straight line parallel to one side of a triangle divides the other two sides in the same ratio.

§ **412.** **Theorem.** If two straight lines are cut by a series of parallel straight lines, the corresponding segments are proportional.

§ **413.** **Theorem.** A line which divides two sides of a triangle in the same ratio is parallel to the third side.

§ **414.** **Problem.** To construct the fourth proportional to three given lines.

§ **415.** **Problem.** To divide a given straight line into parts that are in a given ratio.

§ **416.** **Problem.** To divide a given straight line into parts proportional to any number of given lines.

§ **417.** **Theorem.** The bisector of an angle of a triangle divides the opposite side into two parts which are proportional to the adjacent sides.

SIMILAR TRIANGLES

§ **421.** **Theorem.** Two triangles similar to the same triangle are similar to each other.

§ **422.** **Theorem.** Two triangles are similar if two angles of one are equal respectively to two angles of the other.

§ **423.** **Theorem.** Two right triangles are similar if an acute angle of one is equal to an acute angle of the other.

§ **424.** **Theorem.** Two isosceles triangles are similar if the angle at the vertex, or a base angle, of one equals the corresponding angle of the other.

§ 425. **Theorem.** Two equilateral triangles are similar.

§ 426. **Theorem.** If the corresponding angles of two triangles have their sides respectively parallel, right side to right side and left side to left side, the triangles are similar.

§ 427. **Theorem.** If the corresponding angles of two triangles have their sides respectively perpendicular, right side to right side and left side to left side, the triangles are similar.

§ 428. **Theorem.** The corresponding sides of similar triangles are proportional.

§ 430. **Theorem.** If two triangles have their corresponding sides proportional, they are similar.

§ 431. **Theorem.** If two triangles have an angle of one equal to an angle of the other and the including sides proportional, they are similar.

§ 432. **Theorem.** The areas of two similar triangles are in the same ratio as the squares of any two corresponding sides.

§ 434. . **Theorem.** In any right triangle, if the altitude is drawn from the vertex of the right angle to the hypotenuse:

(1) The two triangles thus formed are similar to the given triangle and to each other.

(2) The altitude is the mean proportional between the segments of the hypotenuse.

§ 435. **Theorem.** Each side about the right angle is the mean proportional between the hypotenuse and its adjacent segment.

§ 436. **Theorem.** The two segments of the hypotenuse and the squares of the two sides are proportionals.

§ 437. **Theorem.** The square of the hypotenuse is to the square of either side as the hypotenuse is to the segment of the hypotenuse adjacent to that side.

§ 438. **Theorem.** The square of the hypotenuse equals the sum of the squares of the two sides.

§ 439. **Theorem.** A perpendicular drawn to a diameter from any point on the circle is a mean proportional between the segments of the diameter.

§ 440. **Problem.** To construct the mean proportional between two given lines.

SIMILAR POLYGONS

§ 442. **Problem.** Construct a polygon similar to a given polygon having given a side corresponding to a side of the given polygon.

§ 443. **Theorem.** The corresponding angles of two similar polygons are equal.

§ **444.** **Theorem.** The corresponding sides of two similar polygons are proportional.

§ **445.** **Theorem.** The perimeters of two similar polygons are in the same ratio as any two corresponding sides.

§ **446.** **Theorem.** The areas of two similar polygons are in the same ratio as the squares of any two corresponding sides.

PROPORTIONAL LINES CONNECTED WITH CIRCLES

§ **447.** **Theorem.** If two chords intersect within a circle, the product of the segments of one is equal to the product of the segments of the other.

§ **450.** **Theorem.** If a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the secant and its external segment.

§ **451.** **Theorem.** If a tangent and a secant are drawn to a circle from an external point, the square of the tangent is equal to the product of the secant and its external segment.

§ **452.** **Theorem.** If two secants are drawn to a circle from an external point, the product of one secant and its external segment is equal to the product of the other secant and its external segment.

Chapter V. Measurement of Circles

§ **458.** **Problem.** To inscribe a regular hexagon in a circle.

§ **459.** **Theorem.** Any equilateral polygon inscribed in a circle is a regular polygon.

§ **460.** **Problem.** To inscribe a square in a circle.

§ **461.** **Theorem.** If a circle is divided into any number of equal parts, the chords joining the successive points of division form a regular inscribed polygon; and the tangents drawn at the successive points of division form a regular circumscribed polygon.

§ **462.** **Theorem.** Tangents drawn to a circle at the vertices of a regular inscribed polygon form a regular circumscribed polygon of the same number of sides.

§ **463.** **Theorem.** Chords drawn from the middle points of the arcs subtended by the sides of a regular inscribed polygon to the adjacent vertices of the polygon form a regular inscribed polygon of double the number of sides.

§ **464.** **Theorem.** The perimeter of a regular inscribed polygon is less than the perimeter of a regular inscribed polygon of double the number of sides, and less than the circumscribed circle.

§ **465. Theorem.** The area of a regular inscribed polygon is less than the area of a regular inscribed polygon of double the number of sides.

§ **466. Theorem.** The area of a regular polygon is less than the area within the circumscribed circle.

POLYGONS AND CIRCLES

§ **469. Problem.** To divide a line in extreme and mean ratio.

§ **470. Problem.** To construct a regular decagon in a given circle.

§ **471. Problem.** To inscribe a regular pentagon in a circle.

§ **472. Problem.** To inscribe in a given circle a regular pentadecagon, or polygon of fifteen sides.

§ **473. Problem.** To circumscribe a circle about any regular polygon.

§ **474. Problem.** To inscribe a circle in any regular polygon.

§ **476. Theorem.** Two regular polygons of the same number of sides are similar.

§ **477. Theorem.** The perimeters of two similar regular polygons are to each other as their radii, and also as their apothems.

§ **478. Theorem.** The area of a regular polygon is equal to half the product of the perimeter by the apothem.

§ **479. Theorem.** The areas of two regular polygons of the same number of sides are to each other as the squares of their radii, and also as the squares of their apothems.

MEASUREMENT OF THE CIRCLE

§ **485. Theorems on Limits.**

(1) If two variables that approach limits are equal for all their successive values, their limits are equal.

(2) If x is a variable having the limit a, and c is a constant not zero, then $cx \rightarrow ca$ and $\dfrac{x}{c} \rightarrow \dfrac{a}{c}$ as $x \rightarrow a$.

§ **488.** The length of a circle is the common limit of the perimeters of regular inscribed and regular circumscribed polygons as the number of sides is indefinitely doubled.

§ **489.** The area of a circle is the common limit of the areas of regular inscribed and regular circumscribed polygons as the number of sides is indefinitely doubled.

§ **490. Theorem.** The apothem of a regular inscribed polygon approaches as a limit the radius of the circle, as the number of sides is indefinitely doubled. Also, the radius of a regular circumscribed polygon approaches as a limit the radius of the circle, as the number of sides is indefinitely doubled.

§ 492. Theorem. Two circumferences have the same ratio as their radii, or as their diameters.

§ 493. Theorem. The ratio of a circumference to its diameter is constant.

§ 495. Theorem. The circumference of any circle is equal to πd, or to $2\pi r$.

§ 496. Theorem. If an arc s is intercepted by a central angle of u degrees the length of the arc is $\dfrac{u}{360}\pi d$.

§ 497. Theorem. The area of a circle is equal to one-half the product of the circumference by the radius.

§ 498. Theorem. The area of any circle is equal to πr^2.

§ 499. Theorem. The area of any circle is equal to $\frac{1}{4}\pi d^2$.

§ 500. Theorem. The areas of two circles are in the same ratio as the squares of their radii or as the squares of their diameters.

§ 508. Theorem. The area of a sector is equal to half the product of its radius by its arc.

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